

Principal problems in Bose-Einstein condensation of dilute gases

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Abstract

A survey is given of the present state of the art in studying Bose-Einstein condensation of dilute atomic gases. The bulk of attention is focused on the principal theoretical problems, though the related experiments are also mentioned. Both uniform and nonuniform trapped gases are considered. Existing theoretical contradictions are critically analysed. A correct understanding of the principal theoretical problems is necessary for gaining a more penetrating insight into experiments with trapped atoms and for their proper interpretation.

Key words: Bose-Einstein condensation; trapped atoms; dilute gases; fluctuations; low-dimensional systems; superfluids; particle correlations; Feshbach resonance; coherent matter waves

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1 Introduction

Bose-Einstein condensation of dilute atomic gases is intensively developing field of research. There exist several reviews and books on the subject, for instance [1–5]. However, both experiment and theory in this field develop so fast that voluminous works are not able to follow the front line of research. Moreover, the amount of information in the field is so huge that even detailed reviews do not include all aspects of the problem. The aim of this survey is to present a brief description of the modern state of the art related to Bose-Einstein condensation of uniform and trapped dilute gases. The main concern will be given to the principal problems in Bose-Einstein condensation. Some of these problems gave rise to controversies. It is, therefore, important to present a critical analysis of the basic theoretical foundations, whose correct understanding is compulsory for the proper interpretation of experiments with trapped Bose-condensed atoms.

Nowadays, there are several tens of laboratories in different countries, where Bose-Einstein condensates are routinely produced. Initially, the condensation of atoms with nonzero spins has been realized, with the total atomic spin $\mathbf{F} = \mathbf{S} + \mathbf{L} + \mathbf{I}$ corresponding to a boson, where \mathbf{S} is electron spin; \mathbf{L} , electron angular momentum; and with \mathbf{I} denoting the nuclear spin. The following dilute atomic gases have been condensed: ^{87}Rb [6], ^7Li [7], ^{23}Na [8], ^{85}Rb [9], ^{41}K [10], ^{133}Cs [11], H [12], and metastable triplet ^4He [13,14]. Recently, the spinless atoms of ^{174}Yb [15] were condensed. There are indications of producing ultracold condensed molecules $^{85}\text{Rb}_2$ [16] formed, by means of Feshbach resonance, in the Bose-Einstein condensate of ^{85}Rb . The condensation of molecules $^6\text{Li}_2$ [17], from the degenerate Fermi gas of ^6Li has been reported. More fully the condensation of molecules will be detailed in the last section below.

In this review, only the systems with Bose-Einstein condensates will be considered. There exists now an intensive activity of studying degenerate trapped Fermi gases. These, however, will not be touched here, except the case when Fermi atoms form Bose molecules that undergo condensation [17]. In the frame of a brief survey, it is impossible to cover all topics related to cold trapped atoms. So, the emphasis will be given to the principal theoretical points in Bose-Einstein condensation of dilute atomic gases.

2 Condensation Temperature

The influence of atomic interactions on the condensation temperature is a rather nontrivial problem, because of which much efforts have been devoted to finding the dependence of the condensation temperature T_c on effective interaction parameters. The interaction in dilute gases is well characterized by the Fermi contact potential

$$\Phi(\mathbf{r}) = 4\pi \frac{\hbar^2 a_s}{m} \delta(\mathbf{r}) ,$$

in which a_s is the s -wave scattering length and m , atomic mass. The gas is dilute when $|\rho a_s^3| \ll 1$, where $\rho \equiv N/V$ is the number density. The problem to be solved is how T_c depends on ρa_s^2 .

This problem turned out to be complicated even for homogeneous dilute gases, since Bose-Einstein condensation is a second-order phase transition, which is governed by long-range correlations. Because of these correlations, simple mean-field approximations are not always applicable.

In an ideal homogeneous Bose gas, the condensation temperature is

$$T_0 = \frac{2\pi\hbar^2}{mk_B} \left[\frac{\rho}{\zeta(3/2)} \right]^{2/3}, \quad (1)$$

where $\zeta(\cdot)$ is the Riemann zeta function, with $\zeta(3/2) \cong 2.612$. What happens when in the ideal gas the atomic interactions are being switched on, while the density is kept fixed? The corresponding change of the condensation temperature can be characterized by the relative difference

$$\frac{\Delta T_c}{T_0} \equiv \frac{T_c}{T_0} - 1. \quad (2)$$

This quantity has been calculated by a variety of methods. Historical references can be found in [5,18,19]. For repulsive interactions, when the scattering length is positive and changes from zero to a finite value a_s , the generally accepted functional dependence of the relative shift (2) on the *diluteness parameter* or *gas parameter*

$$\delta \equiv \rho^{1/3} a_s = \frac{a_s}{a_0}$$

is given by the asymptotic expansion

$$\frac{\Delta T_c}{T_0} \simeq c_1 \delta + (c'_2 \ln \delta + c''_2) \delta^2, \quad (3)$$

as $\delta \rightarrow 0$. The first-order coefficient c_1 depends on nonperturbative correlation effects. Its most accurate value was found by numerical Monte Carlo calculations for three-dimensional lattice O(2) field theory, which gave $c_1 = 1.29 \pm 0.05$ [20,21] and $c_1 = 1.32 \pm 0.02$ [22,23]. The coefficient c'_2 can be computed exactly using perturbation theory [24], which yields

$$c'_2 = -\frac{64\pi\zeta(1/2)}{3[\zeta(3/2)]^{5/3}} = 19.7518.$$

The remaining second-order coefficient c''_2 depends on nonperturbative physics and can also be extracted from numerical lattice simulations of three-dimensional O(2) scalar field theory [23,24], resulting in $c''_2 = 75.7 \pm 0.4$. Summarizing these results, one has

$$c_1 = 1.32 \pm 0.02, \quad c'_2 = 19.7518, \quad c''_2 = 75.7 \pm 0.4.$$

There have also been many attempts to calculate the nonperturbative coefficients c_1 and c''_2 in different approximations. Thus, within the frameworks of Ursell operators and of Green functions [18], the results are $c_1 = 1.9$, $c'_2 = 5.589$, and $c''_2 = 4.9$, which are rather far from the correct values. Renormalization-group calculations give $c_1 = 1.23$ [25] and $c_1 = 1.15$ [26], but c''_2 was not calculated. The $1/n$ -expansion in the n -component field theory has also been used, yielding in the first and second orders, respectively, $c_1 = 2.33$ [27] and $c_1 = 1.71$ [28], but, again, c''_2 was not computed. We shall not enumerate here many other attempts, trying to find the coefficients in the shift (3), but whose results are not sufficiently reliable (see discussion in [18,29]).

The most accurate results, obtained by means of an analytical approach, and being closest to the Monte Carlo numerical simulations, are due to the usage of the optimized perturbation

theory. The main idea of this theory, whose general formulation was first given in [30] (see further references in survey [31]), is the introduction of control functions governing the convergence of the sequence of optimized approximants. Different variants of the optimized perturbation theory are now widely used for various problems [31], in particular, in quantum field theory [32–38]. There are several ways of introducing control functions.

A straightforward method is the introduction of control functions at the initial stage of a calculational procedure, when they can be incorporated into the trial Hamiltonian or Lagrangian, or into the trial wave function or Green function, which the following perturbation theory will be based on [30]. Control functions are defined at each step of the employed perturbation theory so that to make convergent the sequence of the resulting optimized approximants. This method of introducing control functions into the initial Lagrangian of field theory has been used for calculating the critical temperature shift in [39–42], where at the second, third, and fourth orders of the optimized perturbation theory it was found, respectively, $c_1 = 3.06, 2.45, 1.48$ and $c_2'' = 101.4, 98.2, 82.9$. These sequences seemingly converge to the correct numerical Monte Carlo values. The problem of convergence was specially studied in [43–46], where the optimized perturbation theory was combined with $1/n$ -expansion. The series are shown to converge exponentially. The final estimates for the coefficients are $c_1 = 1.19$ and $c_2'' = 84.9$. Thus, the optimized perturbation theory gives the results for both coefficients c_1 and c_2'' that exponentially converge to the Monte Carlo values [39–46].

It is possible to notice that the introduction of control functions into the initial Lagrangian or Hamiltonian leads to a special scaling of the calculated quantities. Using this, a way of introducing control functions, in the frame of the optimized perturbation theory, can be done at the final stage, when one, first, derives divergent series and then transforms them, with control functions being incorporated in the process of this transformation. A particular transformation method has been suggested by Kleinert [47]. This method was invoked for calculating the coefficient c_1 [48–50]. The calculation, first, starts with deriving loop expansions for $(3+1)$ -dimensional field theory. The highest expansion, derived for c_1 , is the seventh-loop expansion [50]

$$c_1(g) \simeq a_1 g + a_2 g^2 + a_3 g^3 + a_4 g^4 + a_5 g^5$$

in powers of an effective coupling parameter, assumed here to be asymptotically small, $g \rightarrow 0$. The actual c_1 corresponds to the limit $c_1(\infty)$ at $g \rightarrow \infty$. As is clear, the above small- g expansion has no sense for $g \rightarrow \infty$. To find an effective limit $c_1(\infty)$, one has to renormalize the above expansion. Such a renormalization can be done by introducing control functions by means of the Kleinert transformation [47] and by defining these control functions from an optimization procedure. This way also leads to an exponentially fast convergence of the sequence of optimized approximants [47]. The coefficients a_k depend on the number of components n . For the two-component field, corresponding to a Bose system, one has [50]

$$\begin{aligned} a_1 &= 0.223286, & a_2 &= -0.0661032, & a_3 &= 0.026446, \\ a_4 &= -0.0129177, & a_5 &= 0.00729073 & (n &= 2). \end{aligned}$$

Employing the Kleinert method, it is necessary to assume that $c_1(\infty)$ is finite. This limit also essentially depends on the choice of the Wegner exponent, characterizing the approach to the limit $c_1(\infty)$. Kastening [49,50] calculated $c_1 = c_1(\infty)$ for two different values of the Wegner exponent. When the latter is defined self-consistently, from the same expansion, then $c_1 = 1.376$;

but if this exponent is fixed, being chosen from some additional arguments, then $c_1 = 1.611$. The average value $c_1 = 1.269$ seems to be close to the Monte Carlo results. To check the method, Kastening [50] has also calculated $c_1(\infty)$ for one-component and four-component fields, for which Monte Carlo simulations are available as well [51]: Thus, for $n = 1$, $c_1 = 1.09 \pm 0.09$ and for $n = 4$, $c_1 = 1.59 \pm 1.10$. For $n = 1$, the expansion coefficients are

$$\begin{aligned} a_1 &= 0.334931, & a_2 &= -0.178478, & a_3 &= 0.129786, \\ a_4 &= -0.115999, & a_5 &= 0.120433 & (n = 1). \end{aligned}$$

The seven-loop optimized approximant [50] is $c_1 = 1.171$ for the self-consistent choice of the Wegner exponent and $c_1 = 0.973$, if the latter is fixed. The average $c_1 = 1.072$ is close to the Monte Carlo value. In the case of the four-component field,

$$\begin{aligned} a_1 &= 0.167465, & a_2 &= -0.0297465, & a_3 &= 0.00700448, \\ a_4 &= -0.00198926, & a_5 &= 0.000647007 & (n = 4). \end{aligned}$$

The two choices of the Wegner exponent lead [50] to $c_1 = 1.648$ and $c_1 = 1.435$, whose average $c_1 = 1.542$ is again in agreement with the Monte Carlo result. The coefficient c_2'' was not calculated by this method.

The accuracy of the optimized perturbation theory can be essentially improved by introducing several control functions. This has recently been done [52] for calculating the coefficients in the temperature shift (3). For the coefficient c_1 , the seven-loop expansion was used, yielding $c_1 = 1.3$ and for c_2'' , the six-loop expansion was employed, giving $c_2'' = 73.46$.

The results of all reliable calculations show that the critical temperature shift (3) is positive and increases with the increasing diluteness parameter. That is, the temperature of Bose condensation grows under increasing density or under stronger interaction. Such a behaviour may look rather strange, though one should not forget that this happens in the dilute limit, when the diluteness parameter $\delta \equiv \rho^{1/3}a_s$ is small. When this parameter is large, as for liquid helium, Monte Carlo simulations show that the critical temperature decreases with growing density [53]. This is in agreement with the existence of the Bogolubov depletion, according to which the condensate density ρ_0 at zero temperature is given by the equation

$$\frac{\rho_0}{\rho} = 1 - \left(\frac{\delta}{\delta_c} \right)^{3/2}, \quad (4)$$

where $\delta \equiv \rho^{1/3}a_s$ is the diluteness parameter and

$$\delta_c \equiv \left(\frac{9\pi}{64} \right)^{1/3} = 0.761618.$$

The Bogolubov depletion formula (4) demonstrates that there exists a critical value δ_c of the diluteness parameter δ , such that the condensate density ρ_0 becomes zero at the critical temperature $T_c = 0$. The Bogolubov approximation, as is known, is valid for weakly nonideal gases, when $\delta \ll 1$. Therefore the critical diluteness parameter δ_c , when the complete depletion occurs, may not be very accurately estimated in this approximation. However Monte Carlo simulations for a

wide diapason of δ show that the Bogolubov depletion formula is qualitatively correct for $\delta \sim 1$ as well, though it is quantitatively accurate up to $\delta \approx 0.1$ [54].

In this way, the Bose-Einstein condensation temperature T_c as a function of the diluteness parameter δ behaves as follows. First, when δ is asymptotically small, T_c grows with increasing δ according to Eq. (3). But then it turns down, becoming a decreasing function of δ . And T_c goes to zero at $\delta \sim 1$. This type of reentrant behaviour in the whole range of δ has been studied [55] in the Bogolubov approximation. The maximum T_c^{max} of T_c was found to occur at $\delta = \delta_{max} \approx 0.25 \delta_c$, so that

$$\frac{T_c^{max}}{T_0} \approx 1.1, \quad \delta_{max} \approx 0.2. \quad (5)$$

Extracting from [54] the behaviour of T_c , as δ tends to δ_c from below, we get

$$\frac{T_c}{T_0} \simeq 2.582 \sqrt{\delta_c - \delta} \quad (\delta \rightarrow \delta_c). \quad (6)$$

Recall that Eqs. (3) and (6) have to do with a homogeneous three-dimensional system.

For trapped gases, the situation is quite different. Let the trapping be realized by the harmonic potential

$$U(\mathbf{r}) = \frac{m}{2} (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2).$$

Strictly speaking, there is no sharp phase transition for a finite number of atoms N . To rigorously define the Bose-Einstein condensation, one should consider $N \rightarrow \infty$. For the gas of N Bose atoms, trapped in the harmonic potential, the thermodynamic limit is defined [2,4] as

$$N \rightarrow \infty, \quad \omega_0 \rightarrow 0, \quad N\omega_0^3 \rightarrow \text{const},$$

which follows from the definition of thermodynamic stability [2], and where the effective frequency is the geometric average

$$\omega_0 \equiv (\omega_x \omega_y \omega_z)^{1/3}.$$

In this limit, the Bose-Einstein condensation temperature of an ideal gas is

$$T_0 = \frac{\hbar \omega_0}{k_B} \left[\frac{N}{\zeta(3)} \right]^{1/3}. \quad (7)$$

For other types of the trapping potential, the condensation temperature will be different [2]. Taking into account the final-size corrections, one has [4] the ideal-gas condensation temperature

$$T'_0 = T_0 - \frac{\zeta(2) \hbar \bar{\omega}}{2 \zeta(3) k_B}, \quad (8)$$

where $\bar{\omega}$ is the arithmetic average

$$\bar{\omega} \equiv \frac{1}{3} (\omega_x + \omega_y + \omega_z).$$

Similarly to the homogeneous case, it is possible to introduce the diluteness parameter for trapped atoms, whose definition, however, is different. For the harmonically trapped gas, the diluteness parameter is given by the ratio

$$\delta \equiv a_s / \lambda_0 \quad (9)$$

of the scattering length a_s to the de Broglie wavelength at the transition temperature

$$\lambda_0 \equiv \sqrt{\frac{2\pi\hbar^2}{mk_B T_0}} = \sqrt{2\pi} l_0 \left[\frac{\zeta(3)}{N} \right]^{1/6},$$

with $l_0 \equiv \sqrt{\hbar/m\omega_0}$ being the oscillator length. Switching on interatomic interactions [56] changes the condensation temperature to

$$\frac{T_c}{T_0} \simeq 1 - \frac{\zeta(2)\overline{\omega}}{2\zeta(3)\omega_0} \left[\frac{\zeta(3)}{N} \right]^{1/3} + c_1\delta + (c'_2 \ln \delta + c''_2) \delta^2, \quad (10)$$

where δ is asymptotically small. In the case of a uniform Bose gas, the first-order shift in T_c , related to the coefficient c_1 , is sensitive to critical fluctuations and so is nonperturbative. In contrast, the first-order shift for a gas trapped in a harmonic potential is calculable using perturbation theory [4]. The second-order logarithmic coefficient c'_2 is also calculable in perturbation theory, but the coefficient c''_2 is not [56]. The coefficient c''_2 can be calculated by involving Monte Carlo lattice simulations in three-dimensional O(2) field theory. The results for these coefficients [56] are

$$c_1 = -3.426032, \quad c'_2 = -\frac{32\pi\zeta(2)}{3\zeta(3)} = -45.856623, \quad c''_2 = -155.$$

The very first thing that catches the eye, comparing the critical temperature shifts (3) and (10), is that for a homogeneous gas all coefficients c_1 , c'_2 , and c''_2 are positive, while for a trapped gas all these coefficients are negative. This makes the principal difference between the uniform and trapped gases. Switching on repulsive interactions in a uniform gas shifts the condensation temperature up, while in a trapped gas this shifts the condensation temperature down. Monte Carlo investigations [54], of a trapped gas at zero temperature in a wide range of the diluteness parameter δ show that the condensation fraction is well described by the Bogolubov depletion formula (4) up to $\delta \sim 0.1$, but for larger $\delta \sim 1$, the depletion formula (4) is only qualitatively correct. It looks like the depletion formula (4) can be extended for high δ as well, but with the critical diluteness δ_c being different from that given by the Bogolubov approximation.

As is seen, there is no the reentrant effect in trapped gases, in contrast to uniform gases, where the condensation temperature first increases and then diminishes as a function of the diluteness parameter. For trapped gases, T_c is monotonically diminishing function of the gas parameter δ .

The formula (10) for the critical temperature of a dilute trapped Bose gas is reliable. There were some claims that the linear in δ shift could be compensated by the finite-size corrections. However, as follows from (10), all shifts have the same negative sign, so there can be no compensation between different terms. In addition, formula (10) has recently been checked for ^{87}Rb and found to be in good quantitative agreement with experimental data [57]. The principal difference in the behaviour of the critical temperature for uniform and trapped gases can be explained by different properties of fluctuations in these systems.

As will be explained in the following section, the uniform ideal gas is an unstable system, with anomalously strong particle fluctuations. Switching on atomic interactions stabilizes the system, making the fluctuations normal. As a result of such a stabilization, the condensation temperature, first, increases according to Eq. (3). But further strengthening interactions leads

to the condensate depletion. This is why the condensation temperature T_c as a function of the diluteness parameter $\rho^{1/3}a_s$ possesses a specific reentrant behaviour [55].

Contrary to the uniform case, the trapped ideal Bose gas is stable, having normal fluctuations. Atomic interactions play only the destructive role, depleting the condensate. Therefore the critical temperature (10) is a monotonely decreasing function of the interaction strength characterized by the parameter a_s/λ_0 defined in Eq. (9).

3 Condensate Fluctuations

Studying fluctuations in statistical systems is important from several points of view. For instance, fluctuations define the stability of the system and its way of reaching the state of thermodynamic equilibrium [58,59].

In the case of Bose-condensed systems, one usually studies the number-of-particle fluctuations because of their nontrivial properties. Since in literature, there are controversies related to this problem, it is necessary to give a careful analysis of the situation.

The number-of-particle fluctuations are characterized by the dispersion

$$\Delta^2(\hat{N}) \equiv \langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2 \quad (11)$$

for the number-of-particle operator

$$\hat{N} = \int \psi^\dagger(\mathbf{r})\psi(\mathbf{r}) d\mathbf{r} ,$$

with $\psi(\mathbf{r})$ being a field operator. The average number of particles N is a statistical average

$$N = \langle \hat{N} \rangle = \int \rho(\mathbf{r}) d\mathbf{r} ,$$

where $\rho(\mathbf{r}) \equiv \langle \psi^\dagger(\mathbf{r})\psi(\mathbf{r}) \rangle$ is the density of particles. Here we keep in mind an equilibrium state, because of which $\rho(\mathbf{r})$ does not depend on time. But $\rho(\mathbf{r})$ depends on the spatial variable \mathbf{r} , when the system is nonuniform.

The dispersion (11) is directly related to the isothermal compressibility

$$\kappa_T \equiv -\frac{1}{V} \left(\frac{\partial V}{\partial P} \right)_{TN} = \frac{1}{\rho} \left(\frac{\partial \rho}{\partial P} \right)_{TN} \quad (12)$$

by the equality

$$\kappa_T = \frac{\Delta^2(\hat{N})}{k_B T \rho N} , \quad (13)$$

whose derivation can be found, e.g., in [60]. A necessary condition for a system to be stable is the semi-positiveness and finiteness of the compressibility, that is, $0 \leq \kappa_T < \infty$. If the compressibility (12) were infinite, this would mean that an infinitesimal fluctuation of pressure P would lead to an immediate collapse or explosion of the system. Therefore, in the thermodynamic limit, the dispersion (11) should behave as

$$\Delta^2(\hat{N}) \simeq \text{const} \cdot N \quad (N \rightarrow \infty) . \quad (14)$$

If $\Delta^2(\hat{N})$ would be proportional to N in a power higher than one, this would mean that the system is unstable. When the stability condition (14) is satisfied, the number-of-particle fluctuations are called normal, but when Eq. (14) is not valid, so that the compressibility (13) diverges in the thermodynamic limit, the fluctuations are termed anomalous.

The dispersion (11) can be presented as

$$\Delta^2(\hat{N}) = \int [R(\mathbf{r}, \mathbf{r}') - \rho(\mathbf{r})\rho(\mathbf{r}')] d\mathbf{r} d\mathbf{r}' \quad (15)$$

through the density-density correlation function

$$R(\mathbf{r}, \mathbf{r}') \equiv \langle \psi^\dagger(\mathbf{r})\psi(\mathbf{r})\psi^\dagger(\mathbf{r}')\psi(\mathbf{r}') \rangle .$$

The latter is related by the equation

$$R(\mathbf{r}, \mathbf{r}') = \rho(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}') + \rho(\mathbf{r})\rho(\mathbf{r}')g(\mathbf{r}, \mathbf{r}')$$

with the pair correlation function

$$g(\mathbf{r}, \mathbf{r}') \equiv \frac{\langle \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r}) \rangle}{\rho(\mathbf{r})\rho(\mathbf{r}')} .$$

Therefore, another representation of the dispersion (15) is

$$\Delta^2(\hat{N}) = N + \int \rho(\mathbf{r})\rho(\mathbf{r}') [g(\mathbf{r}, \mathbf{r}') - 1] d\mathbf{r} d\mathbf{r}' . \quad (16)$$

Note that the pair correlation function possesses the property $g(\mathbf{r}, \mathbf{r}') = g(\mathbf{r}', \mathbf{r})$. One more form for the dispersion (11) can be obtained by recollecting the definition of the structural factor

$$S(\mathbf{k}) \equiv \frac{1}{N} \int [R(\mathbf{r}, \mathbf{r}') - \rho(\mathbf{r})\rho(\mathbf{r}')] e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} d\mathbf{r} d\mathbf{r}' ,$$

which can also be written as

$$S(\mathbf{k}) = 1 + \frac{1}{N} \int \rho(\mathbf{r})\rho(\mathbf{r}') [g(\mathbf{r}, \mathbf{r}') - 1] e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} d\mathbf{r} d\mathbf{r}' .$$

Then, from Eqs. (15) or (16), it follows that

$$\Delta^2(\hat{N}) = NS(0) . \quad (17)$$

All equations above are valid for any nonuniform system. In the particular case of a uniform system, when $\rho(\mathbf{r}) = \rho$ and $g(\mathbf{r}, \mathbf{r}') = g(\mathbf{r} - \mathbf{r}')$, one has the well known expression

$$S(\mathbf{k}) = 1 + \rho \int [g(\mathbf{r}) - 1] e^{-i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r} .$$

Then the dispersion (16) reduces to

$$\Delta^2(\hat{N}) = N \left\{ 1 + \rho \int [g(\mathbf{r}) - 1] d\mathbf{r} \right\} . \quad (18)$$

We see that particle fluctuations are closely related to the behaviour of the pair correlation function as well as to the structural factor, which can be measured by means of light or neutron scattering [61,62]. One more useful relation comes from the expression of the isothermal compressibility (12) through the isothermal sound velocity s , which reads

$$\kappa_T = \frac{1}{\rho m s^2} , \quad s^2 \equiv \frac{1}{m} \left(\frac{\partial P}{\partial \rho} \right)_{TN} .$$

Then, for the zero-vector structural factor, one has

$$S(0) = \rho k_B T \kappa_T = \frac{k_B T}{m s^2} .$$

So that for the dispersion (17), we find

$$\Delta^2(\hat{N}) = \frac{k_B T}{m s^2} N . \quad (19)$$

Note that the same form (19) can be obtained directly from the Bogolubov theory which provides a good first approximation for dilute gases. Really, it is easy to show [2] that in the Bogolubov approximation

$$S(0) = \frac{k_B T}{m c^2} , \quad c = \sqrt{\frac{4\pi\hbar^2}{m^2} \rho a_s} ,$$

with c being the sound velocity of elementary excitations. Hence, we obtain the same form (19), with $s = c$.

Analyzing the relation of the dispersion (18) with the pair correlation function, we may recall that the general behaviour of the latter at large $r \equiv |\mathbf{r}|$ is

$$g(\mathbf{r}) \simeq 1 + \frac{C e^{-r/\xi}}{r^{d-2+\eta}} \quad (\mathbf{r} \rightarrow \infty) ,$$

where C is a constant, ξ is coherence length, d is space dimensionality, and η is the index of anomalous dimension, $0 \leq \eta \leq 1$. Therefore, for any finite ξ , one has $S(0) \sim \text{const}$ and $\Delta^2 \hat{N} \sim N$, that is, the fluctuations are normal, as it should be for any stable equilibrium system. Fluctuations can become anomalous only when the coherence length diverges, $\xi = \infty$, which occurs at the point of a phase transition. Then $S(0) \sim N^{(2-\eta)/3}$ and

$$\Delta^2(\hat{N}) \sim N^{(5-\eta)/3} \quad (\xi = \infty) .$$

With the largest value $\eta = 1$, we have $\Delta^2(\hat{N}) \sim N^{4/3}$. This means that the fluctuations are anomalous, the compressibility $\kappa_T \rightarrow \infty$, the sound velocity $s \rightarrow 0$, and the structural factor $S(0) \rightarrow \infty$. But all this is not a surprise, since the point of a phase transition is the point of instability. Above as well as below this point, the coherence length is finite and fluctuations are normal, which tells that the system is stable. For instance, in the interacting Bose systems [63,64], the coherence length is $\xi \approx \hbar/m s$.

Since particle fluctuations are directly related to the behaviour of the pair correlation function, it is necessary to keep in mind the general properties of the latter, which follow from those of the

second-order density matrix [65]. The pair correlation function satisfies the asymptotic boundary condition

$$g(\mathbf{r}, \mathbf{r}') \rightarrow 1 \quad (|\mathbf{r} - \mathbf{r}'| \rightarrow \infty)$$

and the exact normalization condition

$$\int \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{N(N-1)} g(\mathbf{r}, \mathbf{r}') d\mathbf{r}d\mathbf{r}' = 1 ,$$

which for the uniform case reduces to

$$\frac{1}{V} \int g(\mathbf{r}) d\mathbf{r} = 1 - \frac{1}{N} .$$

In this normalization condition, the thermodynamic limit is assumed.

The fictitious appearance of anomalous fluctuations is often due to the fact that in the process of calculations the general properties of the pair correlation function have been occasionally broken. As an illustrative example, let us adduce the following. From Eq. (18), it is clear that the major contribution to the integral comes from the behaviour of $g(\mathbf{r})$ at large $r \rightarrow \infty$. At large distance, the interparticle interactions vanish. Then, using the Wick decoupling for $\langle \psi^\dagger(\mathbf{r})\psi^\dagger(0)\psi(0)\psi(\mathbf{r}) \rangle$, one could get

$$\langle \psi^\dagger(\mathbf{r})\psi(\mathbf{r}) \rangle \langle \psi^\dagger(0)\psi(0) \rangle + \langle \psi^\dagger(\mathbf{r})\psi(0) \rangle \langle \psi^\dagger(0)\psi(\mathbf{r}) \rangle .$$

The density of a Bose-Einstein condensate, ρ_0 , can be defined as the limit $|\langle \psi^\dagger(\mathbf{r})\psi(0) \rangle| \rightarrow \rho_0$ for $r \rightarrow \infty$. Consequently, for $g(\mathbf{r})$ at large $r \rightarrow \infty$, one would get $1 + N_0^2/N^2$, which would result in the anomalous fluctuations $\Delta^2(\hat{N}) \sim N_0^2$. However, the obtained limit for $g(\mathbf{r})$ is in agreement with the asymptotic boundary condition $g(\mathbf{r}) \rightarrow 1$, as $r \rightarrow \infty$, only for $N_0 \equiv 0$. Hence, actually, there are no anomalous fluctuations in this case. The contradiction has arisen because the Wick decomposition cannot be used for correlated systems with long-range order.

The particle fluctuations below T_0 for the *ideal* uniform Bose gas have been studied in a systematic way [66,67]. In the grand canonical ensemble, the fluctuations of the occupation number \hat{N}_k for the operator of the number-of-particles with momentum \mathbf{k} , such that $|\mathbf{k}| > 0$, behave as $\Delta^2(\hat{N}_k) \sim N^{4/3}$. Hence, these fluctuations are anomalous. Even more anomalous are the fluctuations of the total number of particles $\Delta^2(\hat{N}) = \Delta^2(\hat{N}_0) = N_0^2$, which are caused by the condensate fluctuations. However, as is noted above, such fluctuations are in agreement with the correct properties of the pair correlation function only when $N_0 = 0$. Calculating the dispersion for the number of particles out of the condensate, $\hat{N}_{out} \equiv \sum_{k>0} \hat{N}_k$, one has [66,67] $\Delta^2(\hat{N}_{out}) \sim N^{4/3}$. In the canonical ensemble, when the total number of particles is rigorously fixed, taking into account the equality $\hat{N}_0 = N - \hat{N}_{out}$, one gets $\Delta^2(\hat{N}_0) = \Delta^2(\hat{N}_{out})$, which is given by

$$\Delta^2(\hat{N}_0) = 4.372 \left(\frac{T}{T_0} \right)^2 N^{4/3} , \quad (20)$$

where T_0 is the condensation temperature (1) for the uniform ideal Bose gas. Formula (20) was first derived by Hauge [68]. At all finite temperatures, the condensate fluctuations are anomalous.

Thus, the ideal uniform Bose gas is a rather pathological object, with anomalous fluctuations. It is easy to find its pressure [69], which for $T < T_0$ is

$$P_0 = \zeta(5/2) \frac{k_B T}{\lambda_T^3}, \quad \lambda_T \equiv \sqrt{\frac{2\pi\hbar^2}{mk_B T}}. \quad (21)$$

As is evident, $\partial P_0/\partial \rho = 0$, so that the compressibility (12) diverges, $\kappa_T \rightarrow \infty$, which is in agreement with the anomalous behaviour of the dispersion $\Delta^2(\hat{N})$. Therefore, the ideal Bose gas is unstable below T_0 , though it is stable above this temperature. For a stable equilibrium system, the compressibility must be finite, fluctuations normal, and also, the observable quantities, calculated in different statistical ensembles, must coincide in the thermodynamic limit. Hence, the ideal uniform Bose gas cannot exist as a stable equilibrium system below the temperature of condensation.

The theory of weakly nonideal uniform Bose gas was developed by Bogolubov [70–72]. The dispersion $\Delta^2(\hat{N})$, which he calculated [71], under the conditions $T < T_0$ and $N - N_0 \ll N$, takes the form

$$\Delta^2(\hat{N}) = \frac{k_B T}{mc^2} N, \quad (22)$$

where $c = \sqrt{4\pi\hbar^2 \rho a_s/m^2}$ is the Bogolubov sound velocity. This is in agreement with the general expression (19). The Bogolubov theory [70–72] is based on three points. First, the condensate is described by introducing the condensate field ψ_0 by means of the so-called Bogolubov prescription $\psi = \psi_0 + \tilde{\psi}$ with ψ_0 treated as a nonoperator term and $\tilde{\psi} = \frac{1}{\sqrt{N}} \sum_{k \neq 0} a_k e^{i\mathbf{k} \cdot \mathbf{r}}$ being the field operator of noncondensed atoms. Then, owing to the assumption $N - N_0 \ll N$, all terms of the Hamiltonian containing the factors of $\tilde{\psi}$, or a_k , of orders higher than two, are omitted. And finally, the remaining approximate Hamiltonian is diagonalized by means of the Bogolubov canonical transformation. Since only the terms of up to the second order in $\tilde{\psi}$, or a_k , are retained in this approximation, the same has to be done in calculating all observable quantities. Considering the expressions that are higher than quadratic in $\tilde{\psi}$ would go beyond the accuracy of the Bogolubov approximation and so would be not self-consistent. This was taken into account by Bogolubov [71] when calculating the dispersion (22) using the relation (18).

The weakly nonideal Bose gas possesses the pressure [69]

$$P = P_0 + \frac{2\pi\hbar^2 a_s}{m} \left[\rho^2 + \frac{\zeta^2(3/2)}{\lambda_T^6} \right], \quad (23)$$

where P_0 is the pressure (21) of the ideal gas. From here, the compressibility (12) is $\kappa_T = 1/\rho mc$. Therefore, due to Eq. (13), the particle dispersion coincides with the Bogolubov form (22).

In this way, including atomic interactions makes the fluctuations normal, thus, stabilizing the system. Returning to the ideal gas would mean setting $a_s \rightarrow 0$, because of which $c \rightarrow 0$ and $\kappa_T \rightarrow \infty$, so that the system would become unstable. Equation (22), as is clear, corresponds to the grand canonical ensemble. In canonical ensemble, the fluctuations of the total number of particles are not defined, but one can find the dispersion $\Delta^2(\hat{N}_0)$, for the number of condensed particles, assumed to be equal to $\Delta^2(\hat{N}_{out})$. For a weakly excited interacting gas, condensate fluctuations were found to be normal, that is, $\Delta^2(\hat{N}_0) \sim N$, both in the canonical as well as in the microcanonical ensembles [73].

In the *trapped* ideal Bose gas, particle fluctuations are normal, contrary to those in the uniform case. Politzer [74] calculated $\Delta^2(\hat{N}_0)$ in both canonical and grand canonical ensembles for atoms

trapped inside a harmonically confining potential, the difference between the ensembles vanishing in the thermodynamic limit. The Politzer result is

$$\Delta^2(\hat{N}_0) = \frac{\pi^2}{6\zeta(3)} \left(\frac{T}{T_0}\right)^3 N, \quad (24)$$

where T_0 is the condensation temperature (7) for the harmonically confined ideal gas. Micro-canonical ensemble also displays normal condensate fluctuations [75]. Comparing Eq. (24) with Eq. (20), we see that the anomalous fluctuations of the ideal uniform gas are suppressed by the confining potential.

The influence of interactions in the dilute trapped gas on the size of fluctuations can be understood by remembering that these interactions diminish the condensation temperature according to Eq. (10). Therefore, the dispersion (24) can be transformed to

$$\Delta^2(\hat{N}_0) \approx \frac{\pi^2 N}{6\zeta(3)} \left(\frac{T}{T_0}\right)^3 \left(1 + 10.278 \rho^{1/3} a_s\right). \quad (25)$$

Atomic interactions increase the condensate fluctuations in the trapped gas, leaving the fluctuations normal.

Considering fluctuations in interacting systems, in the frame of the Bogolubov theory [70–72], it is necessary, as is emphasized above, to omit in the course of calculations, all terms containing the products of more than two operators $\tilde{\psi}$, or a_k , corresponding to noncondensed particles. Retaining such higher-order terms would remove the consideration outside the region of applicability of the Bogolubov approach. For example, the dispersion (22), derived in the frame of this approximation, is normal. However, when calculating the dispersion $\Delta^2(\hat{N}_{out})$, one deals with the fourth-order product of the operators a_k , corresponding to the particles excited out of the condensate. The direct calculation of such products, with the standard replacement of the sums over momenta by integration in momentum space, yields infrared divergencies for both uniform as well as trapped gases. These actual divergencies can be formally avoided by considering the discretized sums over the low-energy phonon modes [76], which, anyway, results in anomalous fluctuations $\Delta^2(\hat{N}_{out}) \sim N^{4/3}$ for all interacting systems, trapped or untrapped. This anomalous behaviour arises from the occurrence of infrared divergencies due to the term containing the fourth-order product of operators a_k of noncondensed particles [76–78].

It is, by the way, easy to get the $N^{4/3}$ behaviour of $\Delta^2(\hat{N}_{out})$ without discretizing the low-energy phonon modes. Really, the infrared divergence arising from the improper usage of the fourth-order operator term, which should be omitted in the Bogolubov theory, is caused by the integral $N \int dp/p^2$. For the lower limit of the latter, we could take $p_{min} = \hbar/R$, with the system radius $R = (3V/4\pi)^{1/3}$. Then we would immediately get $\Delta^2(\hat{N}_{out}) \sim N^{4/3}$.

The operator of the total number of particles is the sum $\hat{N} = \hat{N}_0 + \hat{N}_{out}$. Therefore

$$\Delta^2(\hat{N}) = \Delta^2(\hat{N}_0) + \Delta^2(\hat{N}_{out}) + 2 \langle \hat{N}_0 \hat{N}_{out} \rangle - 2N_0 N_{out}.$$

Treating the condensate field as a nonoperator quantity, one has $\hat{N}_0 = N_0 \equiv \langle \hat{N}_0 \rangle$. Then, in the grand canonical ensemble, $\Delta^2(\hat{N}) = \Delta^2(\hat{N}_{out})$. Hence, the anomalous behaviour of the dispersion $\Delta^2(\hat{N}_{out})$ implies the same anomalous behaviour of $\Delta^2(\hat{N})$. The latter would mean that, in the thermodynamic limit, the sound velocity becomes zero, compressibility diverges, $\kappa_T \rightarrow \infty$, and the structural factor $S(0) \rightarrow \infty$ also becomes infinite everywhere below the transition temperature.

Such anomalies have never been observed neither for liquid helium nor for trapped gases. Thus, the excitation spectrum of the trapped Bose-Einstein condensate of ^{87}Rb atoms has been thoroughly investigated, including its long-wavelength limit [79]. The excitation spectrum, sound velocity, and the structural factor were found to be in perfect agreement with the Bogolubov theory predictions. And no anomalies of these quantities were observed. In this way, we have to conclude that the anomalous fluctuations in *interacting* Bose systems, with $\Delta^2(\hat{N}_{out}) \sim N^{4/3}$, are just artifacts caused by a not self-consistent calculational procedure.

The problem of the fictitious appearance of anomalous fluctuations is rather general and occurs for any system with continuous symmetry, when calculations are not self-consistent. Let us consider a system with a Hamiltonian containing the vector operators \mathbf{S} , whose rotation leaves the Hamiltonian unchanged. For concreteness, we may keep in mind the spin operators. The order parameter is $\mathbf{M} \equiv \langle \mathbf{S} \rangle$. An external field \mathbf{h} acts on the operator \mathbf{S} . The influence of the external field on the order parameter is characterized by the tensor of susceptibility $\chi_{\alpha\beta} \equiv \partial M_\alpha / \partial h_\beta$, whose general form can be presented as

$$\chi_{\alpha\beta} = n_\alpha n_\beta \chi_{||} + (\delta_{\alpha\beta} - n_\alpha n_\beta) \chi_{\perp} ,$$

where $\mathbf{n} \equiv \mathbf{h}/h$ and $h \equiv |\mathbf{h}|$. Here $\chi_{||}$ is the longitudinal susceptibility, while χ_{\perp} is the transverse susceptibility. Above the critical temperature T_c , the system is in symmetric phase, so that there are no distinct directions of the order parameter, which implies that $\chi_{||} = \chi_{\perp}$ for $T > T_c$.

But below T_c , the longitudinal and transverse susceptibilities are, in general, different [80]. Usually, the transverse susceptibility, in a model with continuous symmetry, is larger than the longitudinal susceptibility, which results in the dominance of directional over size fluctuations in destroying order [81]. It may even happen that the transverse susceptibility diverges as $\chi_{\perp} \sim h^{-1}$, when $h \rightarrow 0$. This occurs because, under continuous symmetry, all directions of the vector order parameter are equivalent, and an infinitesimally small transverse field may turn the order parameter by a finite angle. Infinite susceptibility means that the considered model is unstable. Such an instability can be easily removed by adding to the model a finite additional field breaking the continuous symmetry. Actually, any real system always has such breaking-symmetry field. For example, the isotropic Heisenberg model is well known to be a cartoon of real magnetic systems, always possessing anisotropic terms due to spin-spin dipole interactions, spin-orbital interactions, indirect exchange, crystalline fields of different nature, and demagnetization anisotropic factors caused by the shape of the sample [82–84]. In the presence of a finite, though may be weak, field h , the transverse susceptibility $\chi_{\perp} \sim h^{-1}$ is finite. The related transverse correlation function $C_{\perp} = \langle \delta S_{\perp} \delta S_{\perp} \rangle$, where $\delta S_{\perp} \equiv S_{\perp} - \langle S_{\perp} \rangle$, has the spatial dependence

$$C_{\perp}(r) \sim \frac{\exp(-r/\xi)}{r^{d-2+\eta}} \quad (r \rightarrow \infty) ,$$

with the correlation length $\xi \sim h^{-1}$. The corresponding Fourier transform reads $C_{\perp}(q) \sim (q^2 + \xi^{-2})^{-1}$, as $q \rightarrow 0$. At zero field $h \equiv 0$, the form $C_{\perp}(q) \sim 1/q^2$ is known as the Goldstone theorem, whose rigorous proof for Bose and Fermi systems was given by Bogolubov [72].

For the longitudinal correlation function $C_{||} = \langle \delta S_{||} \delta S_{||} \rangle$, with $\delta S_{||} \equiv S_{||} - \langle S_{||} \rangle$, one may write [80] an approximate form $C_{||} \sim \langle (\delta S_{\perp})^2 (\delta S_{\perp})^2 \rangle$. Treating the latter in the hydrodynamic approximation, one gets [80] the Wick-type decoupling

$$\langle (\delta S_{\perp})^2 (\delta S_{\perp})^2 \rangle \approx 2 \langle \delta S_{\perp} \delta S_{\perp} \rangle^2 .$$

This gives the relation $C_{\parallel}(r) \sim C_{\perp}^2(r)$, from which it follows that, at zero field $h \equiv 0$, one has $C_{\parallel}(r) \sim 1/r^{2(d-2)}$ and $C_{\parallel}(q) \sim 1/q^{4-d}$. Recall that the index $\eta = 0$ in the hydrodynamic approximation. This tells that the longitudinal susceptibility $\chi_{\parallel} \sim h^{-1/2}$ also diverges at zero field [80]. The character of this divergence follows from the relation $\chi_{\parallel} \sim \int C_{\parallel}(r) d\mathbf{r} \sim N^{(d-2)/3}$, valid for $2 < d < 4$. Then the dispersion of the spin operator \mathbf{S} , for which $\Delta^2(\mathbf{S}) \sim N\chi_{\parallel}$, behaves as $\Delta^2(\mathbf{S}) \sim N^{(d+1)/3}$. In the three-dimensional space, one gets $\Delta^2(\mathbf{S}) \sim N^{4/3}$ for all $T < T_c$, that is, the same kind of anomalous fluctuations as for Bose-Einstein condensates.

If such anomalous fluctuations would really exist, this would mean that there are no stable ordered phases for the systems with broken continuous symmetry. This kind of conclusion would certainly be incorrect. It is well known that stable ordered phases, such as magnetic phases, do exist [82–85]. The longitudinal susceptibility for these phases diverges solely at the critical point T_c , where the correlation length behaves as $\xi \sim a|1 - T/T_c|^{-\nu}$, with a critical exponent ν , and a being the mean interparticle distance. Far below T_c , the correlation length at zero field is always finite, the correlation function exponentially decays, and the longitudinal susceptibility is also finite, hence, the order-parameter fluctuations are normal.

The origin of the fictitious anomalous fluctuations in the systems with continuous symmetry is the same as that occurring in the case of the Bose-Einstein condensate. The hydrodynamic approximation consists in leaving in the Hamiltonian only the terms up to second order in the operators δS_{\perp} . For spin systems, δS_{\perp} plays the role of the magnon operators. To calculate $\Delta^2(\mathbf{S})$, one needs to consider the product of four operators δS_{\perp} , which one treats by invoking the Wick decoupling and using the hydrodynamic approximation. However, the four-order operator products cannot be correctly treated in the second-order hydrodynamic approximation. This is the same as to try to employ a linear approximation for describing strongly nonlinear effects. Because of this inconsistency, there arise infrared divergencies leading to the appearance of fictitious anomalous fluctuations.

The same inconsistency appears if one tries to calculate the four-order terms in the particle dispersion of quantum liquids, such as helium, by using the hydrodynamic approach, which is a second-order approximation. No anomalous fluctuations arise, when calculations are self-consistent, as is demonstrated by Bogolubov [71] when considering the fluctuations in weakly interacting gases with Bose-Einstein condensate.

Let us now summarize the behaviour of fluctuations in Bose-condensed systems. The ideal uniform Bose gas, below the condensation temperature, is an unstable system with anomalous condensate fluctuations described by the dispersion (20). Switching on atomic interactions stabilizes the system making fluctuations normal according to Eq. (22).

The trapped ideal Bose gas is stable, with normal fluctuations, as is described by the dispersion (24) for a harmonic confining potential. Switching on interactions leaves the fluctuations normal, but increases them, as is illustrated by Eq. (25). Different properties of ideal uniform and trapped gases explain why the condensation temperature in a uniform gas displays the reentrant behaviour as a function of the scattering length a_s , while the condensation temperature in a trapped gas is a monotonely decreasing function of a_s .

4 Condensate Growth

Formation of the Bose-Einstein condensed phase from a strongly nonequilibrium, completely disordered, initial state, ultimately leading to long-range ordering, is one of the most interesting and fundamental dynamic problems of the physics of multiparticle systems. The attempts of describing this process have encountered controversies. First, Levich and Yakhot [86], using the semiclassical kinetic equations, concluded that the process of condensation requires infinite time. To make this time finite, it is necessary to assume the existence on the nuclei of the condensed phase in the beginning of the cooling process. Svistunov [87] and Kagan et al. [88–90] also followed a semiclassical approach based on the nonlinear Schrödinger equation. Their analysis of the kinetics starts with the statement that the weakly interacting Bose gas possesses large, relative to unity, occupation numbers and, thus, is essentially a classical field phenomenon. According to their picture [87–90], the condensate formation goes through three stages. At the first stage, substantial suppression of nonequilibrium fluctuations of the density occurs. Then a kind of short- or medium-range order appears, when the quasicondensate is formed, such that the density fluctuations are suppressed, but phase fluctuations are essential. And the true condensate arises only at the last, third, stage, when the long-range order develops, with all fluctuations of density as well as phase being relaxed. Stoof [91,92] stressed the importance of quantum fluctuations at the initial stage.

Bose-Einstein condensate is a coherent system. The development of coherence in a nonequilibrium Bose gas is somewhat analogous to the emergence of coherence in radiating laser-type systems, where the coherent stage is preceded by the quantum stage, with quantum fluctuations triggering the appearance of the coherent component [93,94].

In order to better understand how the condensate is being formed, it is important to recollect the main characteristic length and time scales that are common for all many-particle systems [95–97].

One evident characteristic length is the interaction radius, which can be well represented by a scattering length a_s . This defines the effective particle velocity $v \sim \hbar/ma_s$, with m being particle mass. The *interaction time*

$$\tau_{int} = \frac{a_s}{v} \sim \frac{ma_s^2}{\hbar} \quad (26)$$

describes the time during which two particles strongly interact with each other in the course of mutual scattering. The related interval

$$0 < t < \tau_{int} \quad (\text{dynamic stage}) \quad (27)$$

is called the dynamic stage, when particles move more or less independently from each other, their motion being described by dynamical equations.

The mean free path $\lambda = 1/\rho a_s^2$, where $\rho \sim a^{-3}$, with a being the mean interatomic distance, is the characteristic length between two scattering events. Typical relations are

$$\frac{\lambda}{a_s} \sim \frac{1}{\rho a_s^3}, \quad \frac{\lambda}{a} \sim \left(\frac{a}{a_s}\right)^2.$$

The correlation time or *local-equilibrium time* is

$$\tau_{loc} = \frac{\lambda}{v} \sim \frac{m}{\hbar \rho a_s}. \quad (28)$$

It is connected with the interaction time (26) as

$$\frac{\tau_{int}}{\tau_{loc}} \sim \rho a_s^3.$$

The local-equilibrium time (28) can also be expressed through other characteristic quantities. The healing or coherence length $\xi = \hbar/ms$ is defined through the sound velocity $s \sim \hbar\sqrt{\rho a_s}/m$. The latter is related to the particle velocity v as $s/v \sim \sqrt{\rho a_s^3}$. For the coherence length, one has

$$\frac{\xi}{a_s} \sim \frac{1}{\sqrt{\rho a_s^3}}, \quad \frac{\xi}{a} \sim \sqrt{\frac{a}{a_s}}, \quad \frac{\xi}{\lambda} \sim \sqrt{\rho a_s^3}.$$

The typical energy scale is given by $\varepsilon \sim \hbar^2 \rho a_s / m$. Therefore, for the local-equilibrium time (28) we get

$$\tau_{loc} = \frac{\lambda}{v} \sim \frac{\xi}{s} \sim \frac{\hbar}{\varepsilon}.$$

The temporal interval

$$\tau_{int} < t < \tau_{loc} \quad (\textit{kinetic stage}) \quad (29)$$

is the kinetic stage, during which particle motion can be described by kinetic equations. At this stage, because of mutual interactions, interparticle correlations begin arising.

The existence of the next stage was advanced in [98] (see also [95–97]). Then the overall system consists of parts inside each of which there is some kind of ordering, while different parts are not correlated with each other. The typical size of each part, l_f , is mesoscopic, being in between the mean particle distance a and the linear size of the whole system, L , so that

$$a \ll l_f \ll L. \quad (30)$$

Since these mesoscopic subregions are not correlated with each other, but distributed randomly in space, they are termed heterophase fluctuations. Examples of the latter are magnetic clusters inside a paramagnetic matrix, disordered regions inside a crystal, crystalline formations inside a liquid, superconducting droplets inside a normal metal, subregions with differing local densities, and so on [95]. In particular, these could be separated spatial regions, inside each of which there is a coherent Bose-Einstein condensate, but with no coherence between the different spatial regions. Denoting the average velocity of motion for each heterophase nucleus as v_f , we may introduce the *heterophase fluctuation time*

$$\tau_f = \frac{l_f}{v_f}, \quad (31)$$

which is a characteristic lifetime of a local heterophase fluctuation. During the temporal interval

$$\tau_{loc} \ll t \ll \tau_f \quad (\textit{heterophase stage}), \quad (32)$$

defining the heterophase stage, the system is spatially nonuniform, consisting of subregions with different order parameters [95]. The actual value of τ_f depends on the particular system considered, but it is always larger than the local-equilibrium time τ_{loc} , since to organize a kind of order across the distance $l_f \gg a$ requires a time $\tau_f \gg \tau_{loc}$.

The final temporal interval is the hydrodynamic stage lasting the time defined by the inequalities

$$\tau_f \ll t \ll \tau_{exp} \quad (\text{hydrodynamic stage}) , \quad (33)$$

where τ_{exp} is the experiment time or the observation time. In many cases, one may set $\tau_{exp} \rightarrow \infty$. At this final stage, nonequilibrium systems can be described by hydrodynamic equations.

For usual condense matter, such as solid or liquid, one has $\rho a_s^3 \sim 1$, so that $s \sim v$ and $a_s \sim a \sim \lambda \sim \xi \ll l_f$. Then the kinetic stage (29) is practically absent, and only three stages are left, dynamic, heterophase, and hydrodynamic.

For dilute gases, when $\rho a_s^3 \ll 1$, one has $s \ll v$, so that $a_s \ll a \ll \xi \ll \lambda$. Then all four temporal stages are present, since $\tau_{int} \ll \tau_{loc} \ll \tau_f$. To estimate the characteristic scales for dilute gases, let us take the values typical of experiments with the trapped atoms of ^{87}Rb and ^{23}Na , for which $m \sim 10^{-22}$ g, $a_s \sim 5 \times 10^{-7}$ cm, and $\rho \sim (10^{11} - 10^{15}) \text{ sm}^{-3}$. Then the parameter $\rho a_s^3 \sim 10^{-8} - 10^{-4}$ is really small. Atomic velocity is $v \sim 10$ cm/s and sound velocity is $s \sim (10^{-3} - 10^{-1})$ cm/s. Mean interatomic distance is $a \sim (10^{-5} - 10^{-4})$ cm. Coherence length is $\xi \sim (10^{-4} - 10^{-2})$ cm. Mean free path is $\lambda \sim (10^{-2} - 10^2)$ cm. The interaction time (26) is $\tau_{int} \sim 10^{-8}$ s, that is, the dynamic stage (27) is very short. The local-equilibrium time (28) is $\tau_{loc} \sim (10^{-4} - 1)$ s. Hence, the kinetic stage (29) is much longer than the dynamic stage (27), because of which the latter can be neglected in practical description.

In the frame of the defined time scales, one may consider the formation of the Bose-Einstein condensate from a nonequilibrium disordered state. Then, according to [87–90], we have the following picture. The dynamic stage (27), being rather short, is usually omitted from the consideration. During the kinetic stage (29), substantial suppression of nonequilibrium fluctuations of the density occurs. After the local-equilibrium time τ_{loc} , mid-range order develops in regions of size $l_f \geq \xi$. Different spatial regions are not correlated with each other. Such a situation is typical of heterophase fluctuations. At the heterophase stage (32), density fluctuations are suppressed, but phase fluctuations are essential. The authors [87–90] call such a system *quasicondensate*. After the fluctuation time τ_f , coherence penetrates through the whole system, true long-range order develops, so that the Bose-Einstein condensate is finally formed. At the last hydrodynamic stage (33), the Bose-condensed gas relaxes to equilibrium.

The formation process of a Bose-Einstein condensate in a trap was described by Gardiner et al. [99,100] using a master equation based on quantum kinetic theory. The description assumed the existence of one condensate mode in interaction with an equilibrium thermalized bath of noncondensate atoms at fixed temperature T and chemical potential μ . The number of condensed atoms $N_0(t)$ satisfies the rate equation

$$\frac{dN_0}{dt} = 2W(N_0) \left\{ 1 + \left[1 - \exp\left(\frac{\Delta\mu}{k_B T}\right) \right] N_0 \right\} , \quad (34)$$

in which $\Delta\mu = \mu_c(N_0) - \mu$ is the difference between the condensate chemical potential $\mu_c(N_0)$ and the bath chemical potential μ , while $W(N_0)$ is a transition probability from the bath to the condensate. A collision between a pair of atoms initially in the bath of atomic vapor results in one of the atoms going to the condensate, because of which $W(0)$ is not zero. Thus, with the initial condition $N_0(0) = 0$, at the initial stage, Eq. (34) gives $N_0(t) \simeq 2W(0)t$.

Growth of a Bose-Einstein condensate from thermal vapor was experimentally studied [101] for atoms of ^{87}Rb . It was found that there exists a latency time of $(10^{-4} - 10^{-3})$ s at which

a condensate is first detected. According to the time scales described above, the latency time should correspond to the local-equilibrium time τ_{loc} , since before this time, at the kinetic stage, there cannot be yet any condensate, which appears after τ_{loc} . Then the slow linear growth of the condensate was observed [101] till the time of 5×10^{-3} s. The latter should correspond to the heterophase fluctuation time τ_f . The interval $\tau_{loc} < t < \tau_f$ is the heterophase quasicondensate stage. The numerical calculations based on the rate equation (34) do not reproduce the measured condensate growth data accurately in the quasicondensate regime. This equation describes better the condensate growth at the last hydrodynamic stage, when $t \gg \tau_f$. Then there is exponential growth till the condensate reaches its asymptotic equilibrium value at about 10^{-2} s. At this last stage, instead of Eq. (34), one can use the simple form

$$N_0(t) = N_0(t_0) \{1 - \exp[-\Gamma(t - t_0)]\} , \quad (35)$$

valid for $t_0 \gg \tau_{loc}$, with a fitted relaxation rate Γ .

There is no yet a quantitative theory describing nonequilibrium condensate formation from the very beginning of the process, starting from the kinetic stage, with no condensate, and going through the heterophase and hydrodynamic stages.

5 Low-Dimensional Condensates

The properties of systems in one and two dimensions can essentially differ from those in three dimensions. The Bose systems with the contact interaction

$$\Phi_d(\mathbf{r}) = A_d \delta(\mathbf{r}) , \quad (36)$$

where \mathbf{r} is a d -dimensional vector, have been studied for the spatial dimensionality $d = 1$ and $d = 2$ as well as for the usual three-dimensional case. For the latter, one has $A_s = 4\pi\hbar^2 a_s/m$.

First of all let us recall that the Bose-Einstein condensation in uniform systems does not exist at finite temperatures in $d = 1$ as well as in $d = 2$. This rigorously follows from the Bogolubov theorem [72] as is shown in [102,103].

The features of low-dimensional Bose systems with the interaction (36) can differ owing to the strength of their atomic interactions. This can be illustrated by the classical example of the one-dimensional Bose gas, solved by Lieb and Liniger [104,105]. Let us consider the ground-state energy E , written in the dimensionless form as

$$e(g) \equiv \frac{\hbar^2 n_1^2 E}{2mN} \quad \left(n_1 \equiv \frac{N}{L} \right) ,$$

being a function of the dimensionless coupling

$$g \equiv \frac{mA_1}{\hbar^2 n_1} .$$

In the weak-coupling limit, when $g \ll 1$, the energy $e(g)$ possesses an asymptotic expansion

$$e(g) \simeq g + c_{3/2} g^{3/2} + c_2 g^2 + c_{5/2} g^{5/2} ,$$

with the coefficients [106–108]

$$c_{3/2} = -\frac{4}{3\pi} = -0.424413, \quad c_2 = 0.06535, \quad c_{5/2} = -0.017201.$$

In the strong coupling limit, with $g \rightarrow \infty$, which is also called the Tonks-Girardeau regime [109–111], one has

$$e(\infty) = \frac{\pi^2}{3} = 3.289868.$$

In the Tonks-Girardeau regime, the wave functions of the Bose system are exactly mapped [110] by the Girardeau mapping $\Psi_B = |\Psi_F|$, where Ψ_F are the eigenstates of the ideal one-dimensional Fermi gas provided by the simple Slater determinants. Thermodynamic functions of the one-dimensional system are analytic at any finite temperature [112], in agreement with the absence of a finite-temperature phase transition [102,103].

Though there is no Bose-Einstein condensates in $d = 1, 2$ in finite-temperature uniform systems, the properties of the latter display noticeable change when varying temperature. This can be demonstrated by considering the first-order density matrix

$$\rho_1^{(d)}(\mathbf{r}) \equiv \langle \psi^\dagger(\mathbf{r})\psi(0) \rangle, \quad (37)$$

in which the upper index d shows dimensionality.

For the one-dimensional case, at zero temperature, one finds [113–115] the large-distance decay

$$\rho_1^{(1)}(\mathbf{r}) \sim r^{-\gamma} \quad (T = 0), \quad (38)$$

where $\gamma \equiv mc/2\pi\rho$ and $c = \sqrt{2A_d/m}$ is the sound velocity. While at any finite temperature [115,116], for $r \rightarrow \infty$, one has

$$\rho_1^{(1)}(\mathbf{r}) \sim \exp\left(-\frac{mk_BT}{2n_1} r\right) \quad (T > 0). \quad (39)$$

This means that there is no long-range order at any $T \geq 0$, but at $T = 0$ there arises *mid-range* order characterized by the power-law decay (38). Therefore, at $T = 0$, there can exist superfluidity.

For the two-dimensional uniform gas at zero temperature the long-range order does exist [5], since

$$\rho_1^{(2)}(\mathbf{r}) \rightarrow \rho_0 > 0 \quad (T = 0), \quad (40)$$

as $r \rightarrow \infty$. At finite temperatures, below the Kosterlitz-Thouless temperature T_{KT} , which is of the order of the degeneracy temperature $T_d \sim \hbar^2 \rho^{2/d}/mk_B$, the matrix (37) decays algebraically [117–120],

$$\rho_1^{(2)}(\mathbf{r}) \sim r^{-\gamma} \quad (0 < T < T_{KT}), \quad (41)$$

similarly to the case (38), where $r \rightarrow \infty$. Above the temperature T_{KT} , the density matrix (37) decays exponentially, in analogy with Eq. (39). Therefore, in the two-dimensional uniform Bose systems, there is Bose-Einstein condensate at $T = 0$, and there exists superfluidity at $0 < T < T_{KT}$, though there is no condensate at any finite temperature.

The situation is different for low-dimensional *trapped* gases, for which Bose condensation can occur at finite temperatures. Recall that for a finite number of particles N there is never a rigorous

phase transition, but Bose condensation happens as a gradual crossover, which can be very sharp and practically undistinguishable from the rigorous phase transition, when the number of particles N is sufficiently large. This concerns trapped systems of any finite dimensionality.

Phase transformations of the crossover type can occur, in some cases, even in the thermodynamic limit [95,121]. Therefore, it is useful to generalize the notion of the phase transition, according to the behaviour of the order parameter, as follows [95]. The phase transition is of *first order*, when the order parameter changes, at the transition point, by a jump. The phase transition is of *second order*, when the order parameter at the transition point changes continuously from zero to a finite value. And the phase transition is of *third order*, or of *crossover type*, when the order parameter changes continuously, being nonzero, so that there occurs a qualitative change of its behaviour. The crossover point can be defined as a point separating the qualitatively different regions. Often, the crossover points can be identified with the locations of the extrema of some derivatives [95,121,122].

Bose-Einstein condensation of trapped ideal gases in different dimensions and in various traps has been reviewed in [2]. For $d = 1$, the quasiclassical approximation does not show any condensation [2,123]. However, treating accurately the lower discrete levels displays a crossover transition, which, for a harmonic trap of frequency ω_z , can be located [124] at the temperature

$$T_0^{(1)} = \frac{\hbar\omega_z N}{k_B \ln(2N)} . \quad (42)$$

For a two-dimensional harmonic trap, with the radial frequency ω_\perp , the quasiclassical approach is appropriate yielding [2,123] the crossover temperature

$$T_0^{(2)} = \frac{\sqrt{6N}\hbar\omega_\perp}{\pi k_B} . \quad (43)$$

Below the temperatures (42) or (43) there occurs a macroscopic occupation of the single-particle ground-state energy level, because of which it is permissible to talk on Bose-Einstein condensation [2]. The crossover temperature T_c in low-dimensional trapped dilute gases with interaction is of the order of the temperatures (42) and (43).

Dealing with interacting gases, it is not always easy to define the single-particle spectrum, from which one would like to separate out the ground-state level ascribed to Bose condensate [2]. Nevertheless, in many cases, the condensate density can be defined, serving as a reliable order parameter. A more general approach could be by introducing the *order indices* [65,125]. An order index, for a reduced p -particle density matrix $\hat{\rho}_p$, is defined [125] as

$$\omega_p \equiv \frac{\ln \|\hat{\rho}_p\|}{\ln |\text{Tr} \hat{\rho}_p|} , \quad (44)$$

where $\|\cdot\|$ implies the Hermitian norm. If the total number of particles N is large, one has $\ln |\text{Tr} \hat{\rho}_p| \simeq p \ln N$. For a pure Bose condensate, $\omega_p = 1$, while $\omega_p = 0$ when there is no condensate. And in the intermediate region, the order index (44) varies in the interval $0 < \omega_p < 1$, being equal to $\omega_p = \ln N_0 / \ln N$ [125].

Although in reality there are no exactly one- or two-dimensional systems, it is possible, by varying the shape of the trapping potentials, to produce dilute cold gases in highly anisotropic configurations, where the motion of atoms is quenched in one or two directions. The radial

frequency ω_\perp and the axial frequency ω_z of a cylindrical harmonic trap define, respectively, the transverse and longitudinal oscillator lengths

$$l_\perp \equiv \sqrt{\frac{\hbar}{m\omega_\perp}}, \quad l_z \equiv \sqrt{\frac{\hbar}{m\omega_z}}.$$

When the trap is strongly squeezed in the radial direction, so that $l_\perp \ll l_z$, one has the quasi-one-dimensional configuration. And if the trap is squeezed in the axial direction, so that $l_z \ll l_\perp$, one gets the quasi-two-dimensional system. These two opposite configurations correspond to the cigar-shape or disc-shape traps. For such highly anisotropic traps, it is possible to reduce the generic three-dimensional consideration to an effective one- or two-dimensional case, with the appropriate interaction constant A_d in the potential (36). If the scattering length a_s is smaller, by modulus, than all oscillator lengths, the reduction to a low-dimensional case is straightforward, requiring just to integrate out the variables related to the quenched directions [2]. Thus, for the quasi-one-dimensional trap, one finds

$$A_1 = \frac{2\hbar^2 a_s}{ml_\perp^2} \quad (|a_s| \ll l_\perp \ll l_z). \quad (45)$$

And for the quasi-two-dimensional trap, one obtains

$$A_2 = \sqrt{8\pi} \frac{\hbar^2 a_s}{ml_z} \quad (|a_s| \ll l_z \ll l_\perp). \quad (46)$$

The situation becomes more complicated when the scattering length is comparable with an oscillator length. For example, if the scattering length is of the order or even larger than l_\perp , then in the quasi-one-dimensional case, the quartic self-interaction could be replaced by a nonpolynomial interaction [126–128] or by a sextic term [129,130]. A simple form of A_1 for arbitrary scattering lengths was suggested by Olshanii [131],

$$A_1 = \frac{2\hbar^2 a_s}{ml_\perp^2} \left[1 - \frac{|\zeta(1/2)|a_s}{\sqrt{2} l_\perp} \right]^{-1}, \quad (47)$$

where $\zeta(1/2) \cong -1.46$. In the limit $|a_s| \ll l_\perp$, this reduces to Eq. (45). Bose gases confined in highly elongated harmonic traps were investigated over a wide range of interaction strengths using quantum Monte Carlo techniques [132]. The properties of the Bose gases under tight transverse confinement were found to be well reproduced by a one-dimensional Hamiltonian with the contact interaction (36) and the interaction parameter (47). The latter well represents atomic interactions for arbitrary scattering lengths, positive as well as negative. For $|a_s| \gg l_\perp$, the parameter (47) tends to the unitary limit

$$A_1 \simeq -1.9368 l_\perp \hbar \omega_\perp \quad (|a_s| \gg l_\perp), \quad (48)$$

which does not depend on a_s and is attractive.

In the quasi-two-dimensional case, the effective interaction parameter for arbitrary a_s was considered by Petrov et al. [133,134]. Their results can be presented in the form

$$A_2 = \sqrt{8\pi} \frac{\hbar^2 a_s}{ml_z} \left[1 + \frac{2a_s}{\sqrt{2\pi} l_z} \ln \left(\frac{l_\perp}{\sqrt{2\pi} l_z} \right) \right]^{-1}. \quad (49)$$

In the limit of the asymptotically small scattering length $|a_s| \ll l_z$, Eq. (49) reduces to the form (46). However, under the condition

$$\frac{a_s}{l_z} \ln \frac{l_\perp}{l_z} \gg 1 ,$$

the interaction parameter (49) becomes

$$A_2 \simeq \frac{2\pi\hbar^2}{m \ln(l_\perp/\sqrt{2\pi}l_z)} , \quad (50)$$

and does not depend on a_s . One could also introduce, instead of the contact interaction (36), some more complicated interaction potentials [130,135,136].

Note that for repulsive interactions, when $a_s > 0$, the parameter (49) is always positive. But for attractive interactions, when $a_s < 0$, the effective quasi-two-dimensional interaction (49) has a resonance form, diverging at an intermediate scattering length

$$a_s = - \frac{\sqrt{2\pi} l_z}{2 \ln(l_\perp/\sqrt{2\pi}l_z)} ,$$

and changing sign from negative to positive.

Quasi-one-dimensional strongly elongated cigar-shaped traps have been used in several experiments [137–145]. Quasi-two-dimensional disc-shaped traps have also been employed in experiments [141,146]. For example, in experiments [142,143] the trap frequencies were $\omega_\perp = 2\pi \times 365$ Hz and $\omega_z = 2\pi \times 14$ Hz. The frequency ratio $\omega_\perp/\omega_z = 26$ corresponds to the length ratio $l_z/l_\perp = 5$, which describes an elongated trap. The radial squeezing can be increased [139], reaching $\omega_\perp = 2\pi \times 715$ Hz, which translates into the length ratio $l_z/l_\perp = 7$. Quasi-one-dimensional condensates can also exist in waveguides [147].

6 Crossover Region

In the vicinity of critical points, there always exists a region, where the system experiences strong fluctuations. This region is termed critical. In finite systems, the phase transition is of crossover type. Around the point of crossover, there can arise strong fluctuations, similarly to those appearing in the vicinity of the critical points. Then the region around a crossover point, where fluctuations are strong, can be called crossover region. Bose-Einstein condensation of trapped atoms is always a crossover. In three dimensions, the latter can be rather sharp, when the number of trapped particles is large. Therefore the crossover region can be quite narrow. But in very anisotropic traps, in which the quasi-one-dimensional or quasi-two-dimensional configuration is realized, the condensation crossover can go as a very smeared process, with a wide crossover region.

Considering phase transitions under varying temperature, one usually defines the critical region by means of the Ginzburg temperature T_G , such that in the critical region $T_G < T < T_c$ fluctuations are strong, while essentially below T_G , they are suppressed [148]. In general, one may distinguish the longitudinal and transverse fluctuations of an order parameter. The transverse, or directional, fluctuations often survive much longer, when lowering temperature below T_c , then the longitudinal, or size, fluctuations. Therefore the actual critical region is characterized rather by the transverse than longitudinal fluctuations [81].

In Bose-Einstein condensates, one may also distinguish two types of fluctuations. The size fluctuations of the order parameter correspond to density fluctuations. The transverse fluctuations are usually associated with phase fluctuations. In the vicinity of the condensation temperature T_c , both these fluctuations are strong. Lowering temperature below T_c first diminishes the density fluctuations, while the phase fluctuations remain yet strong. Lowering temperature further, one comes to a point T_f , where phase fluctuations are getting suppressed as well. Thus, the crossover region is defined by the temperature interval $T_f < T < T_c$. For temperatures essentially lower than T_c , but yet much higher than T_f , the system can be described as being spatially divided into subvolumes, each of which being a coherent condensate, but with no coherence between different subvolumes. Such a system is an example of heterophase matter [95,149], in this particular case, of a heterophase condensate. It has also been called quasicondensate [150]. As is discussed in Section 4, the heterophase condensate, or quasicondensate, appears in the process of nonequilibrium condensation. And it may also exist in the crossover region of equilibrium systems. The crossover regions are especially pronounced in low-dimensional trapped gases [133,151,152], though they also occur in purely three-dimensional cases [153].

To consider phase fluctuations, it is necessary, first of all, to introduce a phase operator. In the works on Bose-Einstein condensation, this is commonly done by presenting the field operator ψ as

$$\psi(\mathbf{r}) = \exp \{i\hat{\varphi}(\mathbf{r})\} \sqrt{\hat{\rho}(\mathbf{r})} \quad ? \quad (51)$$

Here $\hat{\varphi}$ is the phase operator and $\hat{\rho} \equiv \psi^\dagger \psi$. The question-mark in Eq. (51) is the sign of warning that one should not blindly trust this equation, which, strictly speaking, is not correct. The problem of introducing the phase operator is a long-standing and not yet completely solved one, as can be inferred from reviews [154–157] (see also a recent discussion in [158]). Thus, in order to satisfy the identity $\hat{\rho} \equiv \psi^\dagger \psi$, the phase operator is to be Hermitian, $\hat{\varphi}^\dagger = \hat{\varphi}$, so that the creation field operator could be written as

$$\psi^\dagger(\mathbf{r}) = \sqrt{\hat{\rho}(\mathbf{r})} \exp \{-i\hat{\varphi}(\mathbf{r})\} \quad ? \quad (52)$$

Here and in what follows, we continue marking by the question-mark those equations that in strict sense are not correct. Above all, the phase operator cannot be defined as a Hermitian operator [154–157].

The fact that the representation (51) is not correct can be demonstrated as follows. From this representation, one has $\exp(i\hat{\varphi}) = \psi \hat{\rho}^{-1/2}$. Then, using the commutation relation

$$[\psi(\mathbf{r}), \hat{\rho}(\mathbf{r}')] = \psi(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}') ,$$

which is valid for any (either Bose or Fermi) field operators, one gets

$$[e^{i\hat{\varphi}(\mathbf{r})}, \hat{\rho}(\mathbf{r}')] = e^{i\hat{\varphi}(\mathbf{r})}\delta(\mathbf{r} - \mathbf{r}') \quad ? \quad (53)$$

From here, after expanding the exponentials, it follows the commutation relation

$$[\hat{\rho}(\mathbf{r}), \hat{\varphi}(\mathbf{r}')] = i\delta(\mathbf{r} - \mathbf{r}') \quad ? \quad (54)$$

This is what one standardly employs announcing the density and phase operators being canonically conjugated. However, the relation (54) is improper. Really, if it were correct, then for the number-of-particle operator $\hat{N} \equiv \int \hat{\rho}(\mathbf{r})d\mathbf{r}$ one would have

$$[\hat{N}, \hat{\varphi}(\mathbf{r})] = i \quad ? \quad (55)$$

The latter equation is already suspicious, with its left-hand side depending on \mathbf{r} and the right-hand side containing no such a dependence. Moreover, Eq. (55) can be reduced to a completely ugly form by taking its matrix element over the states $|n\rangle$ from the number basis, for which $\hat{N}|n\rangle = n|n\rangle$. This yields

$$(n - n') \langle n | \hat{\varphi} | n' \rangle = i \delta_{nn'} \quad ? \quad (56)$$

Setting here $n = n'$ results in a senseless equation $0 = i$.

In this way, the representation (51) cannot possess a rigorous mathematical meaning. Then why people use it? To answer the question, let us note that if, instead of the field operator ψ , we would deal with a classical wave function, then the form $\psi = \sqrt{\rho} e^{i\varphi}$ would be absolutely proper, but with the density ρ and phase φ being also classical functions. This remark suggests that the representation (51) could be accepted as an approximate form for a kind of semiclassical approximation, when the density and phase operators could be treated as almost classical functions. This is equivalent to saying that the density and phase fluctuations above their average values, for which $\delta\hat{\rho} \equiv \hat{\rho} - \langle \hat{\rho} \rangle$ and $\delta\hat{\varphi} \equiv \hat{\varphi} - \langle \hat{\varphi} \rangle$, should be very small. The situation characterized by small fluctuations corresponds to the hydrodynamic approximation.

There could be several equivalent ways of introducing a semiclassical picture. One could treat the evolution equations for the mode operators as semiclassical, in analogy with the semiclassical laser theory [159]. One could start from the Gross-Pitaevskii equation, present the wave function in the form $\psi = \sqrt{\rho} e^{i\varphi}$, and then quantize the classical density and phase [4,160]. Or one could employ from the very beginning the representation (51), but not forgetting that it is only approximately meaningful in the hydrodynamic limit, when all fluctuations are small. In this way, one often treats the density and phase fluctuations as not correlated with each other, permitting their separate averaging. And the phase operators are usually considered as random Gaussian variables, which allows one to invoke, for a function $f(\hat{\varphi})$ of $\hat{\varphi}$, the equality

$$\langle \exp\{f(\hat{\varphi})\} \rangle = \exp \left\{ \frac{1}{2} \langle f^2(\hat{\varphi}) \rangle \right\} .$$

It is worthwhile to emphasize that the phase, as such, is not an observable quantity. What one is able to observe, except the density of atoms, is the density of current, whose operator reads

$$\hat{\mathbf{j}} \equiv - \frac{i\hbar}{2m} \left[\psi^\dagger \nabla \psi - (\nabla \psi^\dagger) \psi \right] ,$$

or the velocity, for which the velocity operator is $\hat{\mathbf{v}} = \hat{\mathbf{j}}/\rho$. For the classical quantities, one would have $\hat{\mathbf{j}} = (\hbar\rho/m)\nabla\varphi$ and $\mathbf{v} = (\hbar/m)\nabla\varphi$. Respectively, instead of talking about unobservable phase fluctuations, one could discuss the current or velocity fluctuations, which can be correctly defined without mentioning the problematic phase operator.

The density operator $\hat{\rho}$ and the operator of current $\hat{\mathbf{j}}$ are connected by the continuity equation

$$\frac{\partial \hat{\rho}}{\partial t} + \text{div} \hat{\mathbf{j}} = 0 ,$$

which is valid for an arbitrary trapping potential and any atomic interactions. Hence, current fluctuations (or phase fluctuations) would, generally, provoke density fluctuations, and vice versa, since they are also connected by the continuity equation

$$\frac{\partial}{\partial t} \delta\hat{\rho} + \text{div} \delta\hat{\mathbf{j}} = 0 .$$

Density fluctuations always excite current fluctuations. However, there can exist current fluctuations that do not initiate density fluctuations. This takes place for the divergenceless fluctuations, for which $\text{div} \delta \hat{\mathbf{j}} = 0$. Hence, there can exist circulating current fluctuations that leave the atomic density undisturbed. This explains why the crossover region is mainly defined by the phase (current) fluctuations, but not by the density fluctuations.

In terms of the observable quantities, we can say that, in the vicinity of the crossover temperature T_c , both the density and current fluctuations are strong. In the part of the crossover region, where $T_f < T \ll T_c$, the density fluctuations can be suppressed, but the current fluctuations are yet essential, which is typical of quasicondensate. And for $T \ll T_f$, true condensate develops.

For quasi-one-dimensional gases, the fluctuations temperature T_f can be estimated [151,152] as

$$T_f^{(1)} \approx \frac{\hbar \omega_z}{k_B} \left(\frac{l_\perp^2}{a_s l_z} \right)^{2/3} N^{1/3}. \quad (57)$$

The characteristic phase-fluctuation length is

$$l_f^{(1)} \approx \frac{\hbar \omega_z}{k_B T} \left(\frac{l_\perp^2 l_z^2}{a_s} \right)^{1/3} N^{2/3}, \quad (58)$$

which should be much less than the length L of the trap.

In quasi-two-dimensional gases, the fluctuation temperature becomes [133]

$$T_f^{(2)} \approx \frac{\hbar \omega_\perp}{k_B} \left(\frac{l_z}{a_s} \right)^{1/2} \frac{\sqrt{N}}{\ln N}, \quad (59)$$

with the fluctuation length

$$l_f^{(2)} \approx \lambda_T \exp \left(\lambda_T^2 \rho^{2/3} \right), \quad (60)$$

where it is assumed that $a_s \ll l_z$ and $\lambda_T \equiv \sqrt{2\pi\hbar^2/mk_B T}$ is the thermal wavelength. Here, it should be that $l_f^{(2)} \ll R$, with R being the trap radius.

The existence of fluctuations in the crossover region of elongated Bose-Einstein condensates has been confirmed experimentally [142–145].

7 Condensates and Superfluids

The relation between Bose-Einstein condensation and superfluidity is a long-standing and not completely understood problem. One often thinks of these two phenomena as being closely related. A detailed description of the historical evolution of views on this problem has recently been done in [2]. A brief and nice discussion can also be found in [161]. The current understanding [2,161] is that Bose-Einstein condensation is *neither necessary nor sufficient* for superfluidity. The latter can occur without Bose-Einstein condensation. Thus, in two-dimensional uniform systems, as is mentioned in Section 5, there can exist superfluidity, though the condensate cannot appear at any finite temperature. Huang [161] gives an example of a Bose system in a random field, when superfluidity is destroyed, even though there is a Bose condensate. Carusotto and Castin [162] find superfluid behaviour in a one-dimensional ring of weakly interacting gas outside the Bose

condensed regime. In the present section, we shall adduce arguments why there can be no simple and general relation between the densities of condensate and of superfluid.

The condensate density

$$\rho_0 = \rho - \rho_{out} \quad (61)$$

is defined as the difference between the total density and that of atoms outside the condensate. The latter density for a uniform gas reads

$$\rho_{out} = \frac{1}{V} \sum_{k \neq 0} n_k, \quad n_k \equiv \langle a_k^\dagger a_k \rangle. \quad (62)$$

For instance, in the Bogolubov approximation at low temperature one finds [70,71,163]

$$\frac{\rho_0}{\rho} \simeq 1 - \frac{8}{3\sqrt{\pi}} \sqrt{\rho a_s^3} - \frac{m(k_B T)^2}{12\hbar^3 \rho c}, \quad (63)$$

where $c = \sqrt{4\pi\hbar^2 \rho a_s}/m$ is the Bogolubov sound velocity.

To find the superfluid density ρ_s , one consider a system boosted with the velocity $\mathbf{v} = \{v^\alpha\}$. In the frame, moving with the same velocity, the Hamiltonian is a functional $H\{\psi\}$ of the field operators ψ . In the laboratory frame, the Hamiltonian $H_v = H\{\psi_v\}$ is the same functional, but of the field operators ψ_v given by the Galilean transformation

$$\psi_v(\mathbf{r}, t) = \psi(\mathbf{r}, t) \exp \left\{ \frac{i}{\hbar} \left(m\mathbf{v} \cdot \mathbf{r} - \frac{mv^2}{2} t \right) \right\}.$$

The superfluid component is interpreted as that part of the system, which nontrivially responds to the velocity boost, with the superfluid density

$$\rho_s \equiv \frac{\rho}{mN} \lim_{v \rightarrow 0} \frac{1}{3} \sum_\alpha \frac{\partial}{\partial v^\alpha} \langle \hat{P}_v^\alpha \rangle_v, \quad (64)$$

where the averaging, with the Hamiltonian H_v , is over the total momentum

$$\hat{\mathbf{P}}_v \equiv \int \psi_v^\dagger(\mathbf{r})(-i\hbar\nabla)\psi_v(\mathbf{r}) d\mathbf{r}.$$

Performing the limit $v \rightarrow 0$ in Eq. (64), one has

$$\rho_s = \rho - \rho_n, \quad (65)$$

with the normal density

$$\rho_n = \frac{\rho\beta}{3mN} \Delta^2(\hat{\mathbf{P}}) \quad \left(\beta \equiv \frac{1}{k_B T} \right) \quad (66)$$

expressed through the momentum dispersion

$$\Delta^2(\hat{\mathbf{P}}) \equiv \langle \hat{\mathbf{P}}^2 \rangle - \langle \hat{\mathbf{P}} \rangle^2.$$

When the total current across the system is absent, then $\langle \hat{\mathbf{P}} \rangle = 0$. Combining Eqs. (61) and (65), one gets the formal relation

$$\rho_s = \rho_0 + \rho_{out} - \rho_n. \quad (67)$$

Note that the derivative in Eq. (64) is a particular case of differentiation with respect to a parameter. In general, if a parameter λ enters an operator $\hat{A} = \hat{A}(\lambda)$ as well as the Hamiltonian $H = H(\lambda)$, then

$$\frac{\partial \langle \hat{A} \rangle}{\partial \lambda} = \langle \frac{\partial \hat{A}}{\partial \lambda} \rangle - \beta \left(\langle \hat{A} \frac{\partial H}{\partial \lambda} \rangle - \langle \hat{A} \rangle \langle \frac{\partial H}{\partial \lambda} \rangle \right) .$$

This formula of parametric differentiation is useful for different particular applications.

To find the normal density (66) one needs to calculate the momentum dispersion, which, for a motionless gas, is $\Delta^2(\hat{\mathbf{P}}) = \langle \hat{\mathbf{P}}^2 \rangle$. This can be expressed [2] as

$$\langle \hat{\mathbf{P}}^2 \rangle = \hbar^2 \int \lim_{r_3 \rightarrow r_1} \lim_{r_4 \rightarrow r_2} \nabla_3 \cdot \nabla_4 [\rho_1(\mathbf{r}_4, \mathbf{r}_3) \delta(\mathbf{r}_1 - \mathbf{r}_2) - \rho_2(\mathbf{r}_3, \mathbf{r}_4, \mathbf{r}_1, \mathbf{r}_2)] d\mathbf{r}_1 d\mathbf{r}_2 , \quad (68)$$

where the first-and second-order density matrices, respectively, are

$$\rho_1(\mathbf{r}_4, \mathbf{r}_3) = \langle \psi^\dagger(\mathbf{r}_3) \psi(\mathbf{r}_4) \rangle ,$$

$$\rho_2(\mathbf{r}_3, \mathbf{r}_4, \mathbf{r}_1, \mathbf{r}_2) = \langle \psi^\dagger(\mathbf{r}_2) \psi^\dagger(\mathbf{r}_1) \psi(\mathbf{r}_3) \psi(\mathbf{r}_4) \rangle .$$

Thus, the definition of the superfluid density (65) involves the second derivatives of the first-and second-order density matrices, while the condensate density (61) requires solely the first-order density matrix. For interacting systems, there is no a simple general relation between the reduced density matrices of different orders (see [65] and recent discussions in [164–168]). This means that it is hardly probable that there could exist a general relation between the condensate density and superfluid density.

For Bose systems with condensate, one usually employs the Bogolubov shift $\psi(\mathbf{r}) = \psi_0 + \tilde{\psi}(\mathbf{r})$, separating the condensate part ψ_0 from the field operator $\tilde{\psi}(\mathbf{r})$ of noncondensed particles. The condensate term ψ_0 is treated as a classical function, which becomes a constant for uniform gases. The field operator of noncondensed particles, for which $\langle \tilde{\psi} \rangle = 0$, can be expanded over a basis $\varphi_k(\mathbf{r})$, so that $\tilde{\psi}(\mathbf{r}) = \sum_{k \neq 0} a_k \varphi_k(\mathbf{r})$. For uniform systems, which are considered in what follows, the basis is formed by plane waves $\varphi_k(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} / \sqrt{V}$. In this way, the single-particle density matrix reads

$$\rho_1(\mathbf{r}_1, \mathbf{r}_2) = \rho_0 + \frac{1}{V} \sum_{k \neq 0} n_k e^{i\mathbf{k} \cdot \mathbf{r}_{12}} , \quad (69)$$

where $\mathbf{r}_{12} \equiv \mathbf{r}_1 - \mathbf{r}_2$ and the property $\langle a_k^\dagger a_q \rangle = \delta_{kq} n_k$ is taken into account.

Constructing the second-order density matrix in the Bogolubov approach, one has to omit all terms containing the powers higher than those of second order in the noncondensed field operators $\tilde{\psi}$ or a_k with $k \neq 0$. Then, keeping in mind the momentum conservation, resulting in the condition $\langle a_k a_q \rangle = \delta_{-kq} \langle a_k a_{-k} \rangle$, one obtains

$$\begin{aligned} \rho_2(\mathbf{r}_3, \mathbf{r}_4, \mathbf{r}_1, \mathbf{r}_2) = & \rho_0^2 + \\ & + \frac{\rho_0}{V} \sum_{k \neq 0} \left\{ \langle a_k a_{-k} \rangle \left(e^{i\mathbf{k} \cdot \mathbf{r}_{12}} + e^{i\mathbf{k} \cdot \mathbf{r}_{34}} \right) + n_k \left(e^{-i\mathbf{k} \cdot \mathbf{r}_{12}} + e^{-i\mathbf{k} \cdot \mathbf{r}_{14}} + e^{-i\mathbf{k} \cdot \mathbf{r}_{23}} + e^{-i\mathbf{k} \cdot \mathbf{r}_{24}} \right) \right\} . \end{aligned} \quad (70)$$

The diagonal element of the density matrix (70) gives the pair correlation function

$$g(\mathbf{r}_{12}) = \frac{1}{\rho^2} \rho_2(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_1, \mathbf{r}_2) . \quad (71)$$

For the latter, when $N_0 \approx N$, one finds

$$g(\mathbf{r}) = 1 + \frac{2}{\rho} \int [n_k + \langle a_k a_{-k} \rangle] e^{i\mathbf{k} \cdot \mathbf{r}} \frac{d\mathbf{k}}{(2\pi)^3} . \quad (72)$$

From here, one has the equality

$$\int [g(\mathbf{r}) - 1] d\mathbf{r} = \frac{2}{\rho} \lim_{k \rightarrow 0} [n_k + \langle a_k a_{-k} \rangle] ,$$

which, in the Bogolubov approximation [70,71], yields

$$\int [g(\mathbf{r}) - 1] d\mathbf{r} = \frac{1}{\rho} \left(\frac{k_B T}{mc^2} - 1 \right) ,$$

with c being the Bogolubov sound velocity. It is this equation that was obtained by Bogolubov when calculating the particle dispersion (22), with the use of Eq. (18). A similar way can be followed in the calculation of the momentum dispersion (68), which is necessary for finding the normal density (66).

Note that for the ideal gas it is easy to show that $\rho_0 = \rho_s$ (see details in [2]). This result, however, is not of much meaning. First, the ideal Bose gas cannot be superfluid, since it does not satisfy the Landau criterion of superfluidity. Second, the ideal Bose-condensed gas, as is explained in Sec. 3, is an unstable system, having infinite compressibility.

It is also easy to show that, for any system at asymptotically high temperatures, one has $\rho_n = \rho$, hence $\rho_0 = \rho_s = 0$. This immediately follows from the Maxwell-Boltzmann distribution law $n_k \simeq \exp\{\beta(\mu - \omega_k)\}$, where, assuming the parabolic spectrum ω_k , the chemical potential is

$$\mu \simeq -\frac{3}{2} k_B T \ln \frac{mk_B T}{2\pi\hbar^2 \rho^{2/3}} .$$

To illustrate the way of calculating the momentum dispersion (68) for interacting gases, let us resort to the Bogolubov approach [70–72]. Then, the annihilation and creation operators of non-condensed particles in the momentum representation are a_k and a_k^\dagger , with $\mathbf{k} \neq 0$. After omitting in the Hamiltonian all terms of third and fourth order with respect to these operators, the remaining quadratic form is diagonalized by means of the Bogolubov canonical transformation

$$a_k = u_k b_k + v_k b_{-k}^\dagger , \quad a_k^\dagger = u_k b_k^\dagger + v_k b_{-k} ,$$

where the operators b_k correspond to excitations. As a result of this diagonalization, one gets

$$u_k = \frac{1}{\sqrt{1 - B_k^2}} , \quad v_k = \frac{B_k}{\sqrt{1 - B_k^2}} ,$$

where

$$B_k = \frac{\varepsilon_k - \sqrt{\varepsilon_k^2 + m^2 c^4}}{mc^2}$$

and ε_k is the Bogolubov spectrum

$$\varepsilon_k = \left[(c\hbar k)^2 + \left(\frac{\hbar^2 k^2}{2m} \right)^2 \right]^{1/2} ,$$

which can also be presented in the form

$$\varepsilon_k = \left[\left(mc^2 + \frac{\hbar^2 k^2}{2m} \right)^2 - m^2 c^4 \right]^{1/2}.$$

For the coefficient functions u_k and v_k , one has

$$u_k^2 = \frac{\sqrt{\varepsilon_k^2 + m^2 c^4} + \varepsilon_k}{2\varepsilon_k}, \quad v_k^2 = \frac{\sqrt{\varepsilon_k^2 + m^2 c^4} - \varepsilon_k}{2\varepsilon_k},$$

which yields the properties

$$u_k^2 - v_k^2 = 1, \quad u_k v_k = -\frac{mc^2}{2\varepsilon_k}, \quad u_k^2 + v_k^2 = \frac{\sqrt{\varepsilon_k^2 + m^2 c^4}}{\varepsilon_k}.$$

For the momentum distribution of excitations, one finds

$$\nu_k \equiv \langle b_k^\dagger b_k \rangle = \frac{1}{e^{\beta\varepsilon_k} - 1},$$

while for the momentum distribution of atoms, one obtains

$$n_k \equiv \langle a_k^\dagger a_k \rangle = (u_k^2 + v_k^2)\nu_k + v_k^2.$$

For the anomalous average $\langle a_k a_q \rangle = \delta_{-kq} \langle a_k a_{-k} \rangle$, one gets

$$\langle a_k a_{-k} \rangle = u_k v_k (1 + 2\nu_k).$$

Note that the anomalous average is of the same order as n_k .

In the long-wave limit $k \rightarrow 0$, one has $\varepsilon_k \simeq c\hbar k$, so that

$$\nu_k \simeq \frac{1}{\beta\varepsilon_k} - \frac{1}{2} + \frac{\beta\varepsilon_k}{36},$$

$$n_k \simeq \frac{mc^2}{\beta\varepsilon_k^2} + \frac{1}{2\beta mc^2} + \frac{\beta mc^2}{36} - \frac{1}{2}, \quad \langle a_k a_{-k} \rangle \simeq -\frac{mc^2}{\beta\varepsilon_k^2} - \frac{\beta mc^2}{36}.$$

Then the sum of the normal and anomalous averages gives

$$\lim_{k \rightarrow 0} (n_k + \langle a_k a_{-k} \rangle) = \frac{1}{2} \left(\frac{k_B T}{mc^2} - 1 \right).$$

These properties are useful for calculating the momentum dispersion (68), for which one finds

$$\langle \hat{\mathbf{P}}^2 \rangle = \sum_k (\hbar k)^2 n_k - N_0 \lim_{k \rightarrow 0} (\hbar k)^2 \langle a_k a_{-k} \rangle. \quad (73)$$

Introducing the notation for the mean kinetic energy

$$E_{kin} \equiv \frac{1}{N} \sum_k \frac{(\hbar k)^2}{2m} n_k = \frac{\hbar^2}{2m\rho} \int k^2 n_k \frac{d\mathbf{k}}{(2\pi)^3} \quad (74)$$

and taking account of the limit

$$\lim_{k \rightarrow 0} (\hbar k)^2 \langle a_k a_{-k} \rangle = -mk_B T ,$$

one comes to the expression of the normal density (66), which is

$$\rho_n = \frac{2E_{kin}}{3k_B T} \rho + \frac{\rho_0}{3} . \quad (75)$$

Unfortunately, the mean kinetic energy (74) possesses an ultraviolet divergence, since at $k \rightarrow \infty$, one has $\varepsilon_k \simeq (\hbar k)^2/2m$ and $n_k \simeq (mc/\hbar k)^4$. Consequently, the normal density and, hence, the superfluid density (65) are not defined in the Bogolubov approximation, while the condensate density (61) is perfectly defined.

The message of this section is that a condensate and a superfluid are not synonyms but are rather different things that should not be confused with each other.

8 Collective Excitations

Generally, collective excitations in dilute Bose gases can be of two types: elementary excitations and topological modes. *Elementary excitations* represent small oscillations above the ground state. They are described by the linear Bogolubov-de Gennes equations. In uniform gases, this yields the famous Bogolubov spectrum (see the previous Sec. 7). Elementary excitations leave the average spatial density unchanged. Contrary to this, *topological modes* are strong excitations essentially changing the spatial density distribution. These are described by the nonlinear Gross-Pitaevskii equation. In uniform systems, the known example of a topological mode is a vortex.

In nonuniform trapped gases, both these types of excitations are also present, though the situation becomes much richer. Elementary excitations in trapped gases can be subdivided in two kinds: continuous modes and discrete modes.

When the excitation wavelengths are much shorter than the trap sizes, the excitation spectrum is a continuous function of its wave vector. This is why such excitations can be called *continuous modes*. The measurement of the k dependence of the excitation spectrum for the trapped condensate of ^{87}Rb atoms was accomplished in the nice experiment by Steinhauer et al. [79]. The measured excitation spectrum was found to be in very good agreement with the Bogolubov spectrum in the local density approximation, even close to the long-wavelength limit of the region of applicability. The static structure factor $S(k)$ was also measured and found to agree well with the Bogolubov theory.

A Monte Carlo computation of the static structure factor $S(k)$ of a dilute Bose-Einstein condensate predicts a roton peak in $S(k)$ occurring at $k \approx 8\pi/a_s$, where a_s is the s -wave scattering length [169]. This peak is not steep enough to produce a minimum in the excitation spectrum ε_k .

For excitations with wavelengths $2\pi/k$ which are comparable to the radius of the condensate, the excitation spectrum is characterized by discrete shape-dependent oscillatory modes. Because of this, such excitations can be termed *discrete modes*. For these modes, momentum is not anymore a good quantum number and consequently the elementary excitations are to be classified by the discrete quantum numbers related to the trap symmetry [2,4]. The excitation spectrum of discrete modes is mainly defined by the shape of the trapping potential, essentially not depending on whether bosons or fermions are trapped [170].

Discrete modes have been observed in several experiments. Performing a small modulation of the trapping potential resulted in the observation of center-of-mass oscillations [171–174]. Higher multipolar-order surface oscillations were observed [175]. Scissors modes were also studied in experiments [176,177]. More details can be found in the review [2].

Theoretical calculations at zero temperature agree well with the observed frequencies of discrete modes [2–4]. Temperature dependence of the frequencies and of the corresponding attenuations, due to interactions between the condensate collective modes and noncondensed thermal atoms, have been theoretically analysed for uniform [178] and trapped [179–182] gases. Finite-temperature calculations are in good agreement with experiments for the lowest modes with axial angular momentum quantum numbers $m = 0$ and $m = 2$ [183,184] and for scissors modes [185,186].

The standard way of treating Bose-condensed systems is based on the Bogolubov prescription $\psi = \psi_0 + \tilde{\psi}$, with $\psi_0 \equiv \langle \psi \rangle$, which breaks gauge symmetry. Then a Green function approach is involved for either real-time or imaginary-time Green functions, which are analytical continuations of each other on the complex-time plane [187]. Nonequilibrium processes can be considered by means of either Keldysh [91,188] or Kadanoff-Baym techniques [189,190].

Another approach is based on the Girardeau-Arnoult canonical transformation

$$\alpha_k = (\hat{N}_0 + 1)^{-1/2} a_0^\dagger a_k ,$$

in which a_0 is the condensate operator and \hat{N}_0 is the condensate number-of-particle operator [191,192]. This approach was followed in [181,193,194] and led to an accurate description of the lowest discrete modes [184]. Note that Bogolubov [70,71] also used a similar transformation $\alpha_k = N_0^{-1/2} a_0^\dagger a_k$, with the difference that here N_0 is not an operator but the number of condensed atoms. In both methods, a well developed condensate is assumed to exist and in both the cases there appear the anomalous averages $\langle a_k a_q \rangle$. The resulting equations are also similar in both these approaches.

A very popular trick is when, after breaking gauge symmetry by means of the Bogolubov prescription, one omits all anomalous averages. One usually ascribes this trick to Popov. However, this is neither a correct approximation nor has any relation to Popov. Really, as is easy to infer from the original works by Popov [115,195,196], he never rejected the anomalous averages. Vice versa, he emphasized the necessity of accurately taking them into account in order to correctly calculate self-energies and to avoid nonphysical singularities after breaking gauge symmetry by the Bogolubov prescription.

Moreover it is straightforward to show that, generally, the anomalous averages are of the same order as the normal ones, so that there is no any reasonable justification of omitting the former though leaving the latter [2,197,198]. For example, in the Bogolubov approach (see Sec. 7) the normal average is

$$\langle a_k^\dagger a_k \rangle = \frac{\sqrt{\varepsilon_k^2 + m^2 c^4}}{\varepsilon_k} \left(\frac{1}{2} + \nu_k \right) - \frac{1}{2} ,$$

where ε_k is the Bogolubov spectrum and $\nu_k \equiv (e^{\beta \varepsilon_k} - 1)^{-1}$, while the anomalous average is

$$\langle a_k a_{-k} \rangle = - \frac{m c^2}{\varepsilon_k} \left(\frac{1}{2} + \nu_k \right) .$$

These averages are closely related usually being of the same order. Thus, in the long-wave limit $k \rightarrow 0$, one has $\langle a_k^\dagger a_k \rangle \simeq mc^2/\beta\varepsilon_k^2$ and $\langle a_k a_{-k} \rangle \simeq -mc^2/\beta\varepsilon_k^2$. That is, in this case, the absolute values of the normal and anomalous averages coincide at all temperatures below T_c . As soon as the anomalous averages appear, they must be considered together with the normal averages. There is no any justification of blindly throwing them out.

Then one may ask: if the trick of throwing out anomalous averages is wrong, why, being used by so much people, it often yields reasonable results? For instance, it leads to accurate calculated frequencies of lower collective excitations in the temperature interval $0 < T < 0.6T_c$, though it is not correct for the $m = 2$ mode above $0.6T_c$. The answer to this question is as follows.

For a system with Bose-Einstein condensate, it is possible to employ a representation, based on *mixed coherent states* [60,199], where the gauge symmetry is not broken and, hence, the anomalous averages do not appear at all [125]. In this approach the field operator is presented as

$$\psi(\mathbf{r}, t) = \eta(\mathbf{r}, t)e^{i\alpha} + \tilde{\psi}(\mathbf{r}, t), \quad (76)$$

where $\eta(\mathbf{r}, t)$ is a coherent field and $\alpha \in [0, 2\pi]$ is a random phase. The average of an operator \hat{A} is defined by

$$\langle \hat{A} \rangle \equiv \int_0^{2\pi} \text{Tr} \hat{\rho} \hat{A} \frac{d\alpha}{2\pi}, \quad (77)$$

with $\hat{\rho}$ being a statistical operator. Because of the averaging over the random phase, all anomalous averages are identically zero, so that $\langle \psi \rangle = \langle \tilde{\psi} \rangle = 0$ as well as $\langle \psi\psi \rangle = \langle \tilde{\psi}\tilde{\psi} \rangle = 0$ and so on. The equation for the coherent field $\eta = \eta(\mathbf{r}, t)$ reads

$$i\hbar \frac{\partial \eta}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + U + A_s (|\eta|^2 + 2\tilde{\rho}) - \mu \right] \eta, \quad (78)$$

where $U = U(\mathbf{r}, t)$ is a trapping (external) potential, $A_s \equiv 4\pi\hbar^2 a_s/m$, and $\tilde{\rho} = \tilde{\rho}(\mathbf{r}, t)$ is the density of noncondensed atoms. Equation (78) reminds a generalized Gross-Pitaevskii equation, but it should be stressed that Eq. (78) is an *exact* equation, with no approximation yet being involved. The density of noncondensed atoms can be found from the related single-particle Green function

$$\tilde{G}(12) \equiv -i \langle \hat{T} \tilde{\psi}(1) \tilde{\psi}^\dagger(2) \rangle,$$

in which \hat{T} is the chronological operator and each number j implies a set \mathbf{r}_j, t_j . The evolution equation for \tilde{G} includes the corresponding two-particle Green function \tilde{G}_2 and self-energy

$$\tilde{\Sigma}(12) = A_s \left[2\delta(12)|\eta(1)|^2 + i \int \tilde{G}_2(1113) \tilde{G}^{-1}(32) d(3) \right].$$

This allows us to write the equation for \tilde{G} as

$$\left(i\hbar \frac{\partial}{\partial t_1} + \frac{\hbar^2}{2m} \nabla_1^2 - U + \mu \right) \tilde{G}(12) - \int \tilde{\Sigma}(13) \tilde{G}(32) d(3) = \delta(12), \quad (79)$$

which is also an exact equation. These equations (78) and (79), being exact, at the same time contain no anomalous averages. More details can be found in Ref. [125]. Linearizing the coherent-field equation (78) with respect to the deviation $\delta\eta \equiv \eta - \eta_0$, where η_0 corresponds to the ground

state, it is possible to come to the Bogolubov-de Gennes type equations describing elementary collective excitations.

In the stationary situation, when the external field $U = U(\mathbf{r})$ does not depend on time, the coherent field can be presented as

$$\eta(\mathbf{r}, t) = \sqrt{N_0} \varphi(\mathbf{r}) \exp \left\{ -\frac{i}{\hbar} (E + \mu)t \right\} .$$

Then, Eq. (78) reduces to the stationary eigenvalue problem

$$\hat{H}[\varphi]\varphi(\mathbf{r}) = E\varphi(\mathbf{r}) , \quad (80)$$

with the effective nonlinear Hamiltonian

$$\hat{H}[\varphi] \equiv -\frac{\hbar^2}{2m} \nabla^2 + U + A_s (N_0 |\varphi|^2 + 2\tilde{\rho}) .$$

The eigenfunctions φ_n of Eq. (80) are termed *topological coherent modes* [200–203]. The mode, with the lowest energy E_0 , describes the standard Bose-Einstein condensate, while the modes, with higher energies $E_n > E_0$, correspond to excited nonground-state condensates [200–203]. The generation of topological coherent modes can be realized by applying alternating fields with frequencies in resonance with the desired transition frequencies $\omega_{mn} \equiv (E_m - E_n)/\hbar$. The most known example of a topological mode is a vortex, which can also be excited by means of resonant external fields [204]. Reviews on vortices in trapped condensates can be found in [205,206]. A dipole topological mode was generated in experiment [207]. The properties of different topological modes were studied in several theoretical papers [200–203,207–221]. Such modes can also be generated by a resonant modulation of the scattering length [222–224]. The topological modes can be excited in single-well potentials, in optical lattices [225], and in other multiwell potentials [226,227]. Adiabatic preparation and evolution of two tunneling modes was investigated [228–230]. Thermodynamics of an ensemble of the coherent modes were described [231].

To stress more the principal difference between elementary excitations and topological modes, let us consider the single-particle density matrix $\hat{\rho}_1$, which can be presented as an expansion over its eigenfunctions $|n\rangle$, called natural orbitals [65], so that $\hat{\rho}_1 = \sum_n N_n |n\rangle\langle n|$, where the eigenvalue $N_n \equiv \langle a_n^\dagger a_n \rangle$ is the occupation number. In the case of a system with the usual Bose-Einstein condensate, the ground-state occupation number $N_0 \sim N$ becomes macroscopic, while all other levels, corresponding to elementary excitations, are not macroscopically occupied, so that $N_n/N \rightarrow 0$, as $N \rightarrow \infty$ and $n \neq 0$. Contrary to this, for a system with several topological modes, several occupation numbers N_n become macroscopic. This type of condensate is, sometimes, called fragmented [232]. Such a fragmentation in the spectrum of the single-particle density matrix should not be confused with a spatial fragmentation of a complex system [233]. Thus, a system with several topological modes is an example of the fragmented condensate in the sense of the spectral fragmentation [232]. It is also worth noting that the multimode fragmented condensate is, generally, nonequilibrium, since its realization requires an action of alternating external fields or a special initial preparation.

9 Particle Correlations

Trapped atoms usually form a very dilute gas, such that $\rho a_s^3 \sim 10^{-8} - 10^{-3}$. The quantity ρa_s^3 can be increased either by increasing the density ρ or the scattering length a_s . The latter can

be varied by manipulating external magnetic fields close to Feshbach resonance [234]. These are collision resonances that occur when the energy of a colliding channel coincides with the energy of a long-lived bound state. When the colliding atoms and the bound state have different magnetic momenta, the resonance condition may be tuned via external magnetic fields exploiting the Zeeman effect [2,235]. If the parameter ρa_s^3 is essentially increased, then interparticle correlations could become important and even the whole picture based on the local interaction, involving just a scattering length, could turn invalid. Correlations become crucial, making the simple mean-field approximation unreliable, already at $\rho a_s \sim 10^{-2}$ [54,236].

Feshbach resonances, first, were observed in Bose-Einstein condensates of ^{85}Rb , near the magnetic field $B_0 = 160$ G with width $\Delta B = -6$ G [237], and of ^{23}Na , near $B_0 = 907$ G with $\Delta B = 1$ G [238]. Then Feshbach resonances were realized in other bosonic atomic species [239–242], as well as in several fermionic atoms [243–246]. The behaviour of the scattering length near a Feshbach resonance at magnetic field B_0 is given by the formula

$$a_s(B) = a_s \left(1 - \frac{\Delta B}{B - B_0} \right), \quad (81)$$

in which $a_s \equiv a_s(\infty)$ is the off-resonance scattering length and ΔB characterizes the width of the resonance. Typical behaviour of the relative scattering length $a_s(B)/a_s$ is illustrated in Fig. 1 for ^{85}Rb , with $B_0 = 156$ G and $\Delta B = -6$ G, and in Fig. 2 for ^{23}Na , with $B_0 = 907$ G and $\Delta B = 1$ G. These figures follow directly from Eq. (81). In the case of ^{85}Rb , it was experimentally possible to change $a_s(B)$ by about three orders of magnitude [237]. But for ^{23}Na , the variation of $a_s(B)$ was limited by one order because of the strong enhancement of inelastic collisions close to the Feshbach resonance leading to an essential depletion of the condensate [238,247]. As is seen in Figures 1 and 2, it is possible not merely vary the scattering-length magnitude but even to change its sign. Such experiments were carried out with ^{85}Rb , when the sign of $a_s(B)$ was fastly changed from positive to negative, which led to the explosion of the condensate [248,249].

When atomic interactions are so strong or density is so large that $\rho a_s^3 > 0.1$, then it becomes crucial to take account of atomic correlations which are well characterized by the pair correlation function $g(\mathbf{r}, \mathbf{r}')$ defined in Sec. 3. In many cases, it is sufficient to consider the uniform approximation $g(\mathbf{r})$ depending on the difference of spatial variables. It may also happen that the contact potential is no longer representing interactions correctly and, instead of the contact potential proportional to $\delta(\mathbf{r})$, one should consider a more realistic form $\Phi(\mathbf{r})$.

From the quantum scattering theory [250] it is known that, generally, the scattering amplitude is expressed through the integrals of the type $\int \varphi_m^*(\mathbf{r}) \Phi(\mathbf{r}) \varphi_n(\mathbf{r}) d\mathbf{r}$, where $\varphi(\mathbf{r})$ is a function characterizing the relative motion of two colliding particles and satisfying the Schrödinger equation

$$\left[-\frac{\hbar^2}{2m_{eff}} \nabla^2 + \Phi(\mathbf{r}) - E \right] \varphi(\mathbf{r}) = 0, \quad (82)$$

in which $m_{eff} \equiv m_1 m_2 / (m_1 + m_2)$ is the reduced mass of the colliding particles with masses m_1 and m_2 . For identical particles, with $m_1 = m_2 \equiv m$, the reduced mass $m_{eff} = m/2$. It is only for the low-energy Born approximation when the scattering amplitude contains just the integral $\int \Phi(\mathbf{r}) d\mathbf{r}$, which reduces to a scattering length.

The simplest way of defining the pair correlation function would be by setting $g(\mathbf{r}) \equiv |\varphi(\mathbf{r})|^2$, with $\varphi(\mathbf{r})$ defined by Eq. (82), but with the boundary condition $|\varphi(\mathbf{r})| \rightarrow 1$ as $r \equiv |\mathbf{r}| \rightarrow \infty$,

instead of the standard normalization condition. Such a definition of $g(\mathbf{r})$ was suggested by Bogolubov [70] and employed for describing quantum crystals [251–257], quantum liquids [258,259], melting-crystallization phase transformation [260], and, generally, for considering quantum statistical systems with strongly singular interaction potentials [261]. The parameter E in Eq. (82) has to be chosen from some additional physical conditions. Thus, it is natural to assume that the pair correlation function $g(\mathbf{r}) = |\varphi(\mathbf{r})|^2$ possesses a maximum at \mathbf{r}_0 where the interaction potential $\Phi(\mathbf{r})$ has a minimum. Then one should set $E = \Phi(\mathbf{r}_0)$. For a short-range repulsive potential, one may set $E = 0$.

For dense systems, it is possible to develop more refined ways of constructing the pair correlation function $g(\mathbf{r})$. It is also possible to find $g(\mathbf{r})$ from experiments [62,262] by measuring the structural factor.

Including interparticle correlations from the very beginning implies that the interaction potential $\Phi(\mathbf{r})$ should enter everywhere together with the pair correlation function $g(\mathbf{r})$, forming the smoothed potential $\bar{\Phi}(\mathbf{r}) \equiv \Phi(\mathbf{r})g(\mathbf{r})$, which plays the role of a pseudopotential. The latter is always finite and integrable, since, when $\Phi(\mathbf{r})$ becomes singular, the correlation function $g(\mathbf{r})$ tends to zero, so that the smoothed potential is always nonsingular, which removes the ultraviolet divergencies appearing in the standard consideration starting with a simple mean-field approximation, such as the Hartree-Fock approximation. An approach, involving at all iteration steps only the smoothed potential $\bar{\Phi}(\mathbf{r})$, was, first, suggested in [263], which has been developed into a refined *correlated iteration theory* [60,264,265]. Here we present only the final set of equations of this theory.

For the single-particle Green function

$$G(12) \equiv -i \langle \hat{T} \psi(1) \psi(2) \rangle ,$$

we may write the Dyson equation

$$G(12) = G_0(12) + \int G_0(13) [\Sigma(32) - \Sigma_0(32)] d(3) , \quad (83)$$

in which $G_0(12)$ is any trial Green function associated with the self-energy $\Sigma_0(12)$. Here the same abbreviated notation as in Sec. 8 is used. The self-energy

$$\Sigma(12) = \pm i \int \bar{\Phi}(13) D(132) d(3) \quad (84)$$

is expressed through the smoothed potential

$$\bar{\Phi}(12) \equiv g(12) \Phi(12) , \quad (85)$$

where $g(12)$ is the pair correlation function, and the doubling function $D(123)$ is given by the form

$$D(123) = \hat{Y} D_0(123) , \quad (86)$$

where the starting doubling function is

$$D_0(123) \equiv \delta(13) G(22) \pm G(12) \delta(23) ,$$

and the operator \hat{Y} is defined below. For generality, Bose as well as Fermi statistics are implied here, with the upper sign being related to Bose, while the lower sign, to Fermi statistics.

An iterative solution of Eqs. (84) and (86) follows the scheme

$$D_k \rightarrow \Sigma_{k+1} \rightarrow \hat{Y}_{k+1} \rightarrow D_{k+1} , \quad (87)$$

where the k -order approximation for the operator \hat{Y} is

$$\hat{Y}_k \equiv 1 + \sum_{p=1}^k \hat{y}_k^p , \quad (88)$$

and the action of the operator \hat{y}_k on any three-point function $f(123)$ is defined as

$$\begin{aligned} \hat{y}_k f(123) &\equiv [1 - g(12)] f(123) + \\ &+ \int G(14) \frac{\delta \Sigma_k(43)}{\delta G(56)} [g(52) f(527) G(76) - G(22) G(56) \delta(67)] d(4567) . \end{aligned} \quad (89)$$

At any approximation order, particle interactions enter always in the form of the smoothed potential (85) renormalized by the pair correlation function $g(12)$. Thus, particle correlations are taken into account from the very first step of the correlated iteration theory [60,264,265].

An analogous procedure can be followed for finding the response functions [60]. For instance, let us consider the response function

$$\chi(123) \equiv \mp i \frac{\delta G(12)}{\delta \mu(3)} , \quad (90)$$

where $\delta \mu(1)$ is a variation of the chemical potential. The poles in the energy representation of the function $\chi(12) \equiv \chi(112)$ describe the spectrum of collective excitations. The response function (90) can be presented as the action of an operator \hat{Z} on the initial function

$$\chi_0(123) \equiv \pm i G(13) G(23) ,$$

corresponding to the random-phase approximation, so that

$$\chi(123) = \hat{Z} \chi_0(123) . \quad (91)$$

An iterative solution of the latter equation is realized according to the scheme

$$\Sigma_k \rightarrow \hat{Z}_k \rightarrow \chi_k , \quad (92)$$

in which the k -order approximation of the operator \hat{Z} is

$$\hat{Z}_k \equiv 1 + \sum_{p=1}^k \hat{z}_k^p , \quad (93)$$

with the action of \hat{z}_k prescribed by the rule

$$\hat{z}_k f(123) \equiv \int G(14) G(52) \frac{\delta \Sigma_k(45)}{\delta G(67)} f(673) d(4567) . \quad (94)$$

Here again, we never meet the bare interaction potential, since the self-energy $\Sigma_k(12)$ contains always only the smoothed potential (85).

Wishing to take an explicit account of the arising Bose-Einstein condensate, we may proceed as in Sec. 8, shifting the total field operator as in Eq. (76) and separating an effective wave equation for the coherent field, as in Eq. (78), from the equation for the Green functions of noncondensed particles. However, a more general way is to present the total Green function as the sum

$$G(12) = -i\eta(1)\eta^*(2) + \tilde{G}(12). \quad (95)$$

Note that the shift (76) is sufficient, though not necessary, for the representation (95). A system, characterized by the Green function (95), contains a coherent part, related to the coherent field η , and a noncoherent part of noncondensed particles corresponding to \tilde{G} . The existence of a coherent part can be detected experimentally [266].

The full many-body theory, accurately taking into account particle interactions and correlations, will become necessary as soon as the parameter ρa_s^3 is not anymore small. As is explained at the beginning of this section, this parameter can be made sufficiently large, up to $\rho a_s^3 \sim 1$, by tuning atomic interactions with an external magnetic field close to a Feshbach resonance. In the vicinity of the latter, there exist bound states, when atoms can form molecules, whose interactions are long-ranged, also requiring to accurately take into account particle correlations [267].

10 Molecular Condensation

Since there exist bound or quasibound states for colliding cold atoms, it has been an important challenge to produce ultracold molecules that could form Bose-Einstein condensates. Such molecules would behave as bosons, independently of whether they are composed of bosons or fermions.

One route for producing ultracold molecules is to form them from Bose atoms in the condensed state. Thus, a two-photon stimulated Raman transition in the condensed ^{87}Rb was used to produce $^{87}\text{Rb}_2$ molecules in a rotational-vibrational state [268], and ultracold molecules of $^{23}\text{Na}_2$ were formed [269] through photoassociation of the condensed ^{23}Na . The molecules of $^{23}\text{Na}_2$ were also produced by means of the Feshbach resonance technique [270], which was used as well for producing the molecules $^{133}\text{Cs}_2$ [271]. Employing the Feshbach resonance was found to be the most convenient way of creating cold molecules, and this technique was used in all experimental works cited below. Molecules are formed by sweeping an external magnetic field through the Feshbach resonance, adiabatically converting atoms to molecules.

The detection of whether the created molecules form a Bose-Einstein condensate is a demanding problem requiring to probe coherence properties of the assembly of atoms plus molecules. It was found that the molecules $^{85}\text{Rb}_2$ do form the condensate [16] as well as the molecules $^{87}\text{Rb}_2$ [272]. The molecules made of cold Bose atoms are in highly vibrationally excited states and usually undergo fast decay during 10^{-4} s.

Molecules can also be composed of cold Fermi atoms by tuning the atomic interactions with a varying magnetic field close to a Feshbach resonance. It is possible to vary the s -wave scattering length a_s from positive to negative over many orders of magnitude. This opens up the possibility of investigating the crossover from a molecular condensate to atomic superconductivity [273,274].

The condensate limit occurs for a positive scattering length $a_s > 0$, much smaller than the interatomic spacing, so that $\rho a_s^3 \ll 1$. Then the fermions form weakly bound dimers of a size about a_s . The superconducting limit is expected to occur for a negative scattering length $a_s < 0$, when the distance between fermions in a superconducting pair is much larger than the interatomic spacing. Then the Fermi momentum k_F is such that $k_F |a_s| \ll 1$. The Fermi temperature of a harmonically trapped gas is $T_F \sim (\hbar\omega_0/k_B)(3N)^{1/3}$. The superconducting transition is expected to happen when $T/T_F \sim 10^{-2} - 10^{-1}$. Thus, if $T_F \sim 10^{-6}$ K, one needs to reach the temperatures $T \sim 10^{-7}$ K, which is achievable in modern experiments.

The molecules ${}^6\text{Li}_2$, composed of cold Fermi atoms ${}^6\text{Li}$, were formed near a Feshbach resonance [275–277], being rather long-lived, with a lifetime around 1 s. The molecules ${}^{40}\text{K}_2$ live shorter, about 10^{-3} s [278]. It has been possible to convert to molecules up to 80% of atoms.

Molecules, formed from Fermi atoms, behave themselves as bosons and, hence, can be condensed at sufficiently low temperatures. Thus, the molecules ${}^6\text{Li}_2$ condense at $T_c = 6 \times 10^{-7}$ K; below this temperature condensate fraction of up to 90% has been observed [17, 279–282]. The molecules ${}^{40}\text{K}_2$ can also be condensed, though their condensate fraction being about 10% [283, 284]. It is especially interesting to study the crossover region around a Feshbach resonance, going from positive scattering lengths to negative ones. Then the molecular condensate transforms into a degenerate Fermi gas of atoms. This crossover region was investigated for ${}^6\text{Li}_2$ [280–282] as well as for ${}^{40}\text{K}_2$ [284]. In the latter paper, condensed molecules of ${}^{40}\text{K}_2$ were observed in the region where the scattering length is negative and no stable ${}^{40}\text{K}_2$ molecules would exist in vacuum. It was claimed [284] that those condensed pairs were not bound into molecules, but merely moved together in a correlated fashion, similar to Cooper pairs of electrons in a superconductor, hence the claim was that the superconductivity of ${}^{40}\text{K}$ atoms had been observed, although the exact nature of these pairs remained unclear. However, theoretical calculations showed [285] that these pairs are not the Cooper pairs but are the same tightly-bound molecules as those existing in the condensate region, where $a_s > 0$. The existence of such molecules in the degenerate Fermi-gas region, where $a_s < 0$, is due to the interaction of these molecules with the unbound atoms of the degenerate Fermi gas. This conclusion was confirmed by the experiment [282] with condensed molecules ${}^6\text{Li}_2$, where the large condensate fraction, up to 80%, was observed in the degenerate-gas region, where no stable ${}^6\text{Li}_2$ molecules could exist in vacuum. The existence of these molecules has been interpreted as being stabilized by the presence of the Fermi gas of unbound atoms [282].

The situation, when bound states, or bound clusters, that cannot exist in vacuum, nevertheless, can perfectly survive inside a sea of their unbound constituents is not unique for trapped atoms [122]. The same picture happens for hadrons that are stable below the point of deconfinement but which would not be stable above this point, if they would be placed in vacuum. Nonetheless, a fraction of hadrons can perfectly survive above the deconfinement point provided they are immersed into quark-gluon plasma [121, 122].

Concluding this review, I would like to repeat what was said in the Introduction. The physics of trapped cold atoms is a very wide and fast developing field. A number of interesting and important subjects are not touched here, such as spinor condensates, dipolar condensates, multicomponent condensates, and condensates in optical lattices. I have concentrated in the present review on the most fundamental, to my understanding, problems. And I hope to cover other topics, not touched here, in another publication.

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Figure Captions

Fig. 1. Relative scattering length $a_s(B)/a_s$ for ^{85}Rb close to the Feshbach resonance at $B_0 = 156$ G with $\Delta B = -6$ G.

Fig. 2. Relative scattering length $a_s(B)/a_s$ for ^{23}Na close to the Feshbach resonance at $B_0 = 907$ G with $\Delta B = 1$ G.



